Problem 28.2

Consider $\varphi^4$ theory of a complex scalar field in $d = 4 - \epsilon$. We want to compute the beta function, and the anomalous dimensions of the mass and field. Previously, we calculated the self-energy of the complex scalar propagator and found $A$ and $B$ in OS renormalization scheme, in $\overline{\text{MS}}$ scheme we choose

$$Z_\varphi = 1 + \mathcal{O}(\lambda^2), \quad Z_m = 1 + \frac{\lambda}{8\pi^2\epsilon} + \mathcal{O}(\lambda^2).$$

While we have computed the vertex correction in real scalar field theory, we need to be careful in applying our results here. Using the Feynman rules of complex scalar field theory, the diagrams contributing to the 1-loop vertex correction are

\begin{align*}
\hline
\text{k}_1 & \to & \ell & \to & \text{k}_3 \\
\text{k}_2 & \to & & \to & \text{k}_4 \\
\hline
\end{align*}

Although the signs of the momenta have changed, this will not affect our result. But the diagrams do have different symmetry factors than those of real scalar $\varphi^4$-theory. Here the $s$-channel still has a symmetry factor of 2, but the $t$ and $u$-channels have symmetry factors of 1. Modifying the necessary factors in the vertex correction, in $\overline{\text{MS}}$ scheme, we choose the counterterm

$$Z_\lambda = 1 + \frac{5}{16\pi^2\epsilon} + \mathcal{O}(\lambda^2).$$

For $G = \sum_n G_n \epsilon^{-n} \equiv \log(Z_\lambda Z^{-2}_\varphi)$, and $M = \sum_n M_n \epsilon^{-n} \equiv \log(Z^{-1/2}_\varphi Z^{-1/2}_m)$ we have

$$G_1 = \frac{5\lambda}{16\pi^2} + \mathcal{O}(\lambda^2), \quad \text{and} \quad M_1 = \frac{\lambda}{16\pi^2} + \mathcal{O}(\lambda^2).$$

Demanding that the bare parameters be independent of $\mu$, we find the $\beta$-function to be

$$\beta(\lambda) \equiv \frac{d\lambda}{d\log \mu} = \lambda G'_1 - \epsilon \lambda = \frac{5\lambda^2}{16\pi^2} - \epsilon \lambda + \mathcal{O}(\lambda^3).$$

and the anomalous dimension of the mass and field to be

$$\gamma_m(\lambda) \equiv \frac{1}{m} \frac{dm}{d\log \mu} = \frac{\lambda}{16\pi^2} + \mathcal{O}(\lambda^2), \quad \gamma_\varphi(\lambda) = 0 + \mathcal{O}(\lambda^2).$$
Problem 28.3

Consider a theory of two real scalar fields \( \varphi \) and \( \chi \) in \( d = 6 - \epsilon \), given as
\[
\mathcal{L} = -\frac{1}{2} Z \varphi \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} Z_m m^2 \varphi^2 + Y \varphi - \frac{1}{2} Z \varphi \partial^\mu \chi \partial_\mu \chi - \frac{1}{2} Z_M M^2 \chi^2 + \frac{1}{6} Z g \mu \epsilon/2 \varphi^3 + \frac{1}{2} Z h \varphi^2 \varphi^2 \chi^2.
\]
We will denote \( \varphi \) as \( \ldots \ldots \) and \( \chi \) as \( \ldots \ldots \). Note that as there are no odd powers of \( \chi \) in \( \mathcal{L} \), there are no diagrams we can construct with one external line, i.e., there is a \( \mathbb{Z}_2 \) symmetry \( \chi \rightarrow -\chi \), so \( \langle \chi \rangle = 0 \). The only tadpole diagram given the Feynman rules is \( \ldots \ldots \ldots \), for which we choose \( Y \varphi \) such that \( \langle \varphi \rangle = 0 \).

a) To find the relevant counterterms in \( \overline{\text{MS}} \) scheme, we need to calculate the divergences in \( \Pi_\varphi(k^2) \), \( \Pi_\chi(k^2) \), \( V_\varphi^3 \), and \( V_\varphi \chi^2 \). We compute the 1-loop correction to the \( \varphi \) propagator as
\[
\hat{\Delta}_\varphi(k^2) = \ldots + \ldots \ldots \ldots + \ldots \ldots \ldots + \ldots \ldots \ldots \ldots + \mathcal{O}(\{g,h\})
\]
The first diagram is the same as in \( \varphi^3 \)-theory, as is the second except with a scalar of mass \( M \) running in the loop, where the symmetry factor of both loop diagrams is 2. Using results from \( \varphi^3 \)-theory, the 1-loop correction to the \( \varphi \) propagator is
\[
\Pi_\varphi(k^2) = -\frac{1}{2(4\pi)^3} \left( \frac{2}{\epsilon} + 1 \right) \left( \frac{1}{6} k^2 + m^2 \right) + \int_0^1 dx \log \frac{\mu^2}{D_1} + \int_0^1 dx \log \frac{\mu^2}{D_2} = -(Z_\varphi - 1) k^2 - (Z_m - 1) m^2,
\]
where we will not be explicit about the \( D \) terms as they are finite. Ignoring finite terms, we find
\[
\Pi_\varphi(k^2) = -\frac{1}{2(4\pi)^3} \left( \frac{2}{\epsilon} + 1 \right) \left( \frac{1}{6} k^2 + M^2 \right) + \int_0^1 dx \log \frac{\mu^2}{D_2} = -(Z_\varphi - 1) k^2 - (Z_m - 1) m^2,
\]
and to cancel the \( 1/\epsilon \) divergence, to leading order, we choose \( Z_\varphi \) and \( Z_m \) to be
\[
Z_\varphi = 1 - \frac{1}{\epsilon} \frac{1}{6(4\pi)^3} (g^2 + h^2), \quad Z_m = 1 - \frac{1}{\epsilon} \frac{1}{(4\pi)^3} \left( g^2 + \frac{M^2}{m^2} h^2 \right).
\]

Now we compute the 1-loop correction to the \( \chi \) propagator
\[
\hat{\Delta}_\chi(k^2) = \ldots + \ldots \ldots \ldots + \ldots \ldots \ldots + \mathcal{O}(\{g,h\})
\]
Here we need to proceed a little more carefully in determining the factors proportional to the divergence. We rewrite the integrand of the 1-loop diagram as
\[
\int_0^1 \frac{dx}{(q^2 + D)^2} = \frac{1}{(\ell^2 + m^2)((\ell^2 + k^2)^2 + M^2)},
\]
where \( q \equiv \ell + x k \) and \( D \equiv x(1-x)k^2 + x M^2 + (1-x)M^2 \). Doing the \( \epsilon \) expansion, the \( \chi \) propagator correction is
\[
\Pi_\chi(k^2) = -\frac{h^2}{(4\pi)^3} \left( \frac{2}{\epsilon} + 1 \right) \int_0^1 dx D - \int_0^1 dx D \log \frac{\mu^2}{D} = -(Z_\chi - 1) k^2 - (Z_M - 1) M^2,
\]
\[
= -\frac{1}{(4\pi)^3} \frac{1}{\epsilon} \left( \frac{1}{2} h^2 k^2 + h^2 M^2 + h^2 m^2 \right) + \text{finite} - (Z_\chi - 1) k^2 - (Z_M - 1) M^2.
\]
choosing $Z_\chi$ and $Z_M$ to cancel the $1/\epsilon$ divergence, to first order, we have

$$Z_\chi = 1 - \frac{1}{\epsilon} \frac{h^2}{3(4\pi)^3}, \quad Z_M = 1 - \frac{1}{\epsilon} \frac{1}{(4\pi)^3} \left(h^2 + \frac{m^2}{M^2} h^2\right).$$

The 1-loop correction to the $\varphi^3$ vertex is

$$V_{\varphi^3} = \text{---} + \raisebox{-1.2em}{\includegraphics[height=1.5em]{phi3loop1}} + \raisebox{-1.2em}{\includegraphics[height=1.5em]{phi3loop2}} + \text{---} + \text{---} + O\left(\{g,h\}^4\right).$$

Again, we can just use the results from $\varphi^3$-theory, making sure to insert the appropriate couplings and mass of the particle running in the loop

$$V_{\varphi^3} = Z_g g + \frac{1}{2} \frac{g^3}{(4\pi)^3} \left(\frac{2}{\epsilon} + \int dF_3 \log \frac{\mu}{D_1}\right) + \frac{1}{2} \frac{h^3}{(4\pi)^3} \left(\frac{2}{\epsilon} + \int dF_3 \log \frac{\mu}{D_2}\right),$$

where again we are not too concerned with the $D$'s. Ignoring the finite pieces, we find

$$V_{\varphi^3} = Z_g g + \frac{1}{(4\pi)^3} \epsilon \left(g^3 + h^3\right) \rightarrow Z_g = 1 - \frac{1}{(4\pi)^3} \epsilon \frac{1}{g} \left(g^3 + h^3\right).$$

Lastly, the 1-loop correction to the $\varphi \chi^2$ vertex is

$$V_{\varphi \chi^2} = \text{---} + \raisebox{-1.2em}{\includegraphics[height=1.5em]{phi-chi2loop1}} + \raisebox{-1.2em}{\includegraphics[height=1.5em]{phi-chi2loop2}} + \text{---} + \text{---} + O\left(\{g,h\}^4\right),$$

using the appropriate couplings and mass of the particle running in the loop

$$V_{\varphi \chi^2} = Z_h h + \frac{1}{2} \frac{g^2 h^2}{(4\pi)^3} \left(\frac{2}{\epsilon} + \int dF_3 \log \frac{\mu}{D_1}\right) + \frac{1}{2} \frac{h^3}{(4\pi)^3} \left(\frac{2}{\epsilon} + \int dF_3 \log \frac{\mu}{D_2}\right),$$

here the $D$'s we would choose, unlike above, are of a different form than those in $\varphi^3$ theory, but unlike the propagator correction, here $D$ does not contribute to the divergent piece, thus we need not worry as $q$ is the same in these diagrams. Ignoring the finite pieces, we find

$$V_{\varphi \chi^2} = Z_h h + \frac{1}{(4\pi)^3} \epsilon \left(gh^2 + h^3\right) \rightarrow Z_h = 1 - \frac{1}{(4\pi)^3} \epsilon \left(gh + h^2\right).$$

We now have all the counterterms in MS scheme.

b) For the bare coupling $g_0 = Z_{\varphi}^{-3/2} Z_g g\tilde{\mu}^{\epsilon/2}$, we define $G(g,h) = \sum_n G_n \epsilon^{-n} \equiv \log Z_{\varphi}^{-3/2} Z_g$, asymptotically expanding the log and matching leading divergences

$$G_1 = \frac{1}{(4\pi)^3} \left(\frac{1}{4} h^2 - \frac{3}{4} g^2 - \frac{h^3}{g}\right).$$

For the bare coupling $h_0 = Z_{\chi}^{-1/2} Z_{\chi}^{-1} Z_h h\tilde{\mu}^{\epsilon/2}$, we define $H(g,h) = \sum_n H_n \epsilon^{-n} \equiv \log Z_{\varphi}^{-1/2} Z_{\chi}^{-1} Z_h$, expanding the log and matching leading divergences

$$H_1 = \frac{1}{(4\pi)^3} \left(\frac{1}{12} g^2 - \frac{7}{12} h^2 - gh\right).$$
Proceeding, we differentiate \( \log g_0 \) and \( \log h_0 \) with respect to \( \log \mu \) and demand they vanish

\[
g \beta_g \frac{\partial G}{\partial g} + g \beta_h \frac{\partial G}{\partial h} + \beta_g + \frac{1}{2} \epsilon g = 0, \quad h \beta_g \frac{\partial H}{\partial g} + h \beta_h \frac{\partial H}{\partial h} + \beta_h + \frac{1}{2} \epsilon h = 0,
\]

where the \( \beta \)-functions are \( \beta_g = \frac{dg}{d \log \mu} \) and \( \beta_h = \frac{dh}{d \log \mu} \). Now we solve for \( \beta_g \) and \( \beta_h \) by solving the system of equations to order \( O(\epsilon^0) \)

\[
\beta_g = -\frac{1}{2} \epsilon g + \frac{1}{2} g \left( \frac{\partial G_1}{\partial g} + \frac{h}{\partial h} \frac{\partial G_1}{\partial h} \right), \quad \beta_h = -\frac{1}{2} \epsilon h + \frac{1}{2} h \left( \frac{\partial H_1}{\partial g} + \frac{h}{\partial h} \frac{\partial H_1}{\partial h} \right).
\]

c) Given the counterterms we computed for the theory, we found \( H_1 \) and \( G_1 \), which give the \( \beta \)-functions of the theory

\[
\beta_g = \frac{1}{(4\pi)^3} \left( -\frac{3}{4} g^3 + \frac{1}{4} h^2 g - h^3 \right), \quad \beta_h = \frac{1}{(4\pi)^3} \left( \frac{1}{12} g^2 h - gh^2 - \frac{7}{12} h^3 \right),
\]
in the \( \epsilon \to 0 \) limit.

d) If we fix \( g \) to be positive, the sign of \( h \) will affect the sign of \( \beta_h \) and \( \beta_g \), making the theory either asymptotically free or IR free. Denoting \( a \equiv h/g \), we write the \( \beta \)-functions as

\[
\beta_g = -\frac{1}{(4\pi)^3} g^3 \left( a^3 - \frac{1}{4} a + \frac{3}{4} \right), \quad \beta_h/h = -\frac{1}{(4\pi)^3} g^2 \left( \frac{7}{12} a^2 + a - \frac{1}{12} \right).
\]

\( \beta_g \) is negative for \( a > -1 \) and \( \beta_h/h \) is negative for \( a > \frac{1}{4}(-6 + \sqrt{43}) \). Therefore, both \( \beta \)-functions are negative for \( h/g > \frac{1}{4}(-6 + \sqrt{43}) \), meaning the theory is asymptotically free. For a theory of two real scalars this could indicate an unstable ground state but more generally, strongly coupled low energy physics, and weakly coupled high energy physics. But if \( -1 < h/g < \frac{1}{4}(-6 + \sqrt{43}) \), then both \( \beta \)-functions are positive and the theory is IR free, with well-behaved low energy physics.