Renormalization Group and $\varphi^4$-theory

Consider $\varphi^4$ theory in $d = 4 - \epsilon$ dimensions,

$$L = -\frac{1}{2} Z_\varphi \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} Z_m m^2 \varphi^2 - \frac{1}{4!} Z_\lambda \lambda \tilde{\mu}^\epsilon \varphi^4,$$

where the renormalized fields and parameters are related to the parameters in the bare Lagrangian as

$$\varphi_0 = Z_\varphi^{1/2} \varphi, \quad m_0 = Z_m^{-1/2} Z_m^{1/2} m, \quad \lambda_0 = \lambda \tilde{\mu}^\epsilon Z_\lambda Z_\varphi^{-2}.$$

Previously, we calculated the 1-loop correction to the propagator and the vertex in $\varphi^4$-theory and found

$$i\Pi(k^2) = i(k^2(1 - Z_\varphi) + m^2(1 - Z_m)) + m^2 \frac{i\lambda}{32\pi^2} \left(\frac{2}{\epsilon} + 1 + \log \frac{\mu^2}{m^2}\right) + \mathcal{O}(\lambda^2), \quad (1)$$

$$iV_4(k_1, k_2, k_3, k_4)/\lambda = -iZ_\lambda + i \frac{\lambda}{32\pi^2} \left(\frac{6}{\epsilon} + \int_0^1 dx \left(\log \frac{\mu^2}{D_s} + \log \frac{\mu^2}{D_t} + \log \frac{\mu^2}{D_u}\right)\right) + \mathcal{O}(\lambda^2).$$

where the $D$’s are just the Feynman parameter given by each of the Mandelstam variables in the $s$, $t$, and $u$ channel contributions.

In OS scheme, we made the choice of counterterm coefficients that rendered $\Pi(k^2)$ and $V_4$ finite, to be the entire correction, both the term divergent in $\epsilon$ and the finite piece. This was to ensure that the exact propagator had the correct pole at $k^2 = -m^2$ with residue one. These were the choices of $A$, $B$, and $C$, for $Z_\varphi = 1 + A$, $Z_m = 1 + B$, and $Z_\lambda = 1 + C$, we made previously.

In $\overline{\text{MS}}$ we choose the $Z$’s to cancel just the divergent powers of $1/\epsilon$ arising in the loop integration

$$Z_\varphi = 1 + \sum_{n=1}^{\infty} \frac{a_n(\lambda)}{\epsilon^n}, \quad Z_m = 1 + \sum_{n=1}^{\infty} \frac{b_n(\lambda)}{\epsilon^n}, \quad Z_\lambda = 1 + \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{\epsilon^n}. $$

In $\varphi^4$-theory, we choose $a_1$, $b_1$, and $c_1$ to cancel the $1/\epsilon$ piece of the vertex and propagator corrections, leaving the finite terms, and find

$$a_1(\lambda) = 0 + \mathcal{O}(\lambda^2), \quad b_1(\lambda) = \frac{\lambda}{16\pi^2} + \mathcal{O}(\lambda^2), \quad c_1(\lambda) = \frac{3\lambda}{16\pi^2} + \mathcal{O}(\lambda^2),$$

where all $a_n$, $b_n$, $c_n$ are all $\mathcal{O}(\lambda^2)$ for $n > 1$. 


We now demand that the bare field $\lambda_0 = \mu^* Z_\lambda Z_\phi^{-2}$ is independent of $\mu$. For convenience, we start by defining $G(\lambda, \epsilon) = \sum_{n=1}^{\infty} G_n(\lambda)/\epsilon^n \equiv \log(Z_\lambda Z_\phi^{-2})$ which has an expansion in powers of $1/\epsilon$, where the leading divergence is

$$G(\lambda, \epsilon) = \frac{1}{\epsilon} (c_1(\lambda) - 2a_1(\lambda)) = \frac{3\lambda}{\epsilon 16\pi^2} + O(\lambda^2).$$

We now demand that $\log \lambda_0 = \log \lambda + \epsilon \log \mu + G$ is independent of $\mu$

$$0 = \frac{d \log \lambda_0}{d \log \mu} = \frac{1}{\lambda} \frac{d \lambda}{d \log \mu} + \epsilon + \frac{d G}{d \lambda} \frac{d \lambda}{d \log \mu} \rightarrow \frac{d \lambda}{d \log \mu} \left(1 + \lambda \frac{d G}{d \lambda}\right) + \epsilon \lambda = 0.$$

Demanding that this be finite in the $\epsilon \to 0$ limit in a renormalizable theory, fixes $G_n$ for $n > 2$. For $\phi^4$-theory, we find the $\beta$-function to be

$$\beta(\lambda) \equiv \frac{d \lambda}{d \log \mu} = \lambda G_1' - \epsilon \lambda = \frac{3\lambda^2}{16\pi^2} - \epsilon \lambda + O(\lambda^3),$$

where $\epsilon \lambda$ vanishes in the $\epsilon \to 0$ limit. Note that it is typically more convenient to define the $\beta$-function as above, where we include the term $\epsilon$ term, for reasons we’ll see later.

Now consider the bare mass $m_0 = Z_\phi^{-1/2} Z_m^{1/2}$, and define $M(\lambda, \epsilon) \equiv \log(Z_\phi^{-1/2} Z_m^{1/2})$, which is, to leading order in $1/\epsilon$,

$$M(\lambda, \epsilon) = \frac{1}{2\epsilon}(b_1(\lambda) - a_1(\lambda)) + O(\lambda^2) = \frac{1}{\epsilon} \frac{\lambda}{32\pi^2} + O(\lambda^2).$$

Demanding that $\log m_0 = \log m + M$ is independent of $\mu$, we find

$$0 = \frac{d \log m_0}{d \log \mu} = \beta_\lambda \frac{d M}{d \lambda} + \frac{1}{m} \frac{d m}{d \log \mu} = (\lambda G_1' - \epsilon \lambda) \left(\frac{1}{\epsilon} \frac{d M_1}{d \lambda} + \ldots\right) + \frac{1}{m} \frac{d m}{d \log \mu} = 0.$$

For a renormalizable theory, terms $1/\epsilon$ and higher must cancel as $dm/d \log \mu$ should be finite in the $\epsilon \to 0$ limit. The anomalous dimension is thus

$$\gamma_m(\lambda) \equiv \frac{1}{m} \frac{d m}{d \log \mu} = \frac{\lambda}{32\pi^2} + O(\lambda^2).$$

Lastly, the propagator is related to the bare propagator as $\tilde{\Delta}_0(k^2) = Z_\phi \tilde{\Delta}(k^2)$. Writing $\log Z_\phi = a_1/\epsilon \ldots = 0$ to leading order in $1/\epsilon$. The anomalous dimension of the field is

$$\gamma_\phi(\lambda) = \frac{1}{2} \frac{d \log Z_\phi}{d \log \mu} = 0 + O(\lambda^2).$$

**Asymptotic Symmetry**

Consider the Lagrangian for a theory with two real scalar fields $\phi_1$ and $\phi_2$,

$$\mathcal{L} = \frac{1}{2} \left((\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2\right) - \frac{\lambda}{4!}(\phi_1^4 + \phi_2^4) - \frac{2\rho}{4!}(\phi_1^2 \phi_2^2).$$

Denoting $\phi_1$ as a solid line and $\phi_2$ as a dashed line, we can simply read off the three types of interactions between four scalars in the theory as
\[ \begin{align*}
\times &= -i\lambda, \\
\bullet &\quad -i\lambda, \\
\cdot &\quad -i\rho/3
\end{align*} \]
as there is only a symmetry of 4 in the \( \rho \) vertex.

We can now proceed to compute the \( \lambda \) vertex correction with \( \phi_1 \) external lines (though it doesn’t matter which). This is precisely the same as the 1-loop correction to the vertex in \( \varphi^4 \)-theory, except that now we have \( s, t, \) and \( u \) contributions from both \( \phi_1 \) and \( \phi_2 \) running in the loop. Namely,

Recall from \( \varphi^4 \)-theory that a 1-loop diagram, the first \( s \)-channel for example, gives the contribution

\[
\frac{1}{2} (-i\lambda)^2 \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{i} \tilde{\Delta}((\ell + k_1 + k_2)^2) \frac{1}{i} \tilde{\Delta}(\ell^2) = i \frac{\lambda^2}{16\pi^2} \left( \frac{1}{\epsilon} + \frac{1}{2} \int_0^1 dx \log \frac{\mu^2}{D_s} \right),
\]

where here \( D_s = -s(1-x)x \) as the theory is massless, but this is not important as we are not concerned with the finite piece given by the integral. Adding the \( s, t, \) and \( u \) contributions from the \( \phi_1 \) running in the loop, with \( \lambda \) vertices, and the contributions from the \( \phi_2 \) running in the loop, with \( \rho \) vertices, we find

\[
iV_4 = -iZ_\lambda \lambda + i \frac{3i}{16\pi^2\epsilon} \left( \lambda^2 + \frac{\rho^2}{9} \right) + \text{finite},
\]

and thus the counterterm we choose is, to leading order in \( 1/\epsilon \),

\[
Z_\lambda = 1 + \frac{1}{\epsilon} \frac{3}{16\pi^2} \left( \lambda^2 + \frac{\rho^2}{9} \right).
\]

Now we want to renormalize the \( \rho \) vertex. We find the \( s, t, \) and \( u \) contributions by fixing the external lines, in this case two \( \phi_1 \)'s and two \( \phi_2 \)'s, and seeing what diagrams we can construct given our Feynman rules. There are two \( s \)-channel contributions, corresponding to \( \phi_1 \) and \( \phi_2 \) running in the loop, one \( t \)-channel and one \( u \)-channel contribution, each with a mixed loop of \( \phi_1 \) and \( \phi_2 \)

There is one subtlety here, when we computed the loop integrals in \( \varphi^4 \)-theory and again above in Eq.(2), we had a factor of \( 1/2 \) from the symmetry factor of the diagram, i.e. exchanging the propagators in the loop. In the last two diagrams above, this is no longer a symmetry of the diagram, so when using the results from \( \varphi^4 \), we have to make sure we account for this factor. In all, we have

\[
iV_4 = -iZ_\rho \rho + \frac{i}{16\pi^2\epsilon} \left( \frac{\lambda \rho}{3} + \frac{\lambda \rho}{3} + 2 \frac{\rho^2}{9} + 2 \frac{\rho^2}{9} \right) + \text{finite},
\]

and thus the counterterm at leading order in \( 1/\epsilon \) is

\[
Z_\rho = 1 + \frac{1}{\epsilon} \frac{1}{16\pi^2} \left( \frac{2\lambda \rho}{3} + \frac{4\rho^2}{9} \right).
\]
Lastly, there is no wavefunction renormalization from the corrections to either scalar two-point function at 1-loop order, the self-energy \( \Pi(k^2) \) Eq. (1) is trivial in the massless theory, and therefore \( Z_{\phi_1} = 1 + \mathcal{O}(\lambda^2) \) and \( Z_{\phi_2} = 1 + \mathcal{O}(\lambda^2) \).

We can now compute the \( \beta_\lambda \) and \( \beta_\rho \). Demanding that \( \lambda_0 = \lambda \tilde{\mu} Z_{\phi_1}^{-2} \) be independent of scale, we define \( G_\lambda = \log Z_{\phi_1}^{-2} \), and find that

\[
\frac{d \log \lambda_0}{d \log \mu} = 0 \quad \rightarrow \quad \beta_\lambda \left( 1 + \lambda \frac{\partial G_\lambda}{\partial \lambda} \right) + \epsilon \lambda + \beta_\rho \lambda \frac{\partial G_{\phi_1}}{\partial \rho} = 0
\]

Defining \( G_\rho = \log Z_{\phi_2}^{-1} Z_{\phi_1}^{-1} \), we demand that \( \rho_0 = \rho \tilde{\mu} Z_{\phi_2}^{-1} Z_{\phi_1}^{-1} \) be independent of scale

\[
\frac{d \log \rho_0}{d \log \mu} = 0 \quad \rightarrow \quad \beta_\rho \left( 1 + \rho \frac{\partial G_\rho}{\partial \rho} \right) + \epsilon \rho + \beta_\lambda \rho \frac{\partial G_\rho}{\partial \lambda} = 0.
\]

We can now find the \( \beta \)-functions using \( Z_\lambda \) and \( Z_\rho \) above, where the leading order terms in \( 1/\epsilon \) give \( G_\lambda \) and \( G_\rho \). Solving the system of equations by excessively expanding to leading order in \( 1/\epsilon \), we find that

\[
\beta_\lambda = \frac{3}{16\pi^2} \left( \lambda^2 + \frac{\rho^2}{9} \right) - \lambda \epsilon, \quad \beta_\rho = \frac{1}{8\pi^2} \left( \lambda \rho + \frac{2\rho^2}{3} \right) - \rho \epsilon.
\]

b) Now we compute the \( \beta_{\rho/\lambda} \) simply using the chain rule

\[
\beta_{\rho/\lambda} = \frac{\lambda}{\lambda^2} (\lambda \beta_\rho - \rho \beta_\lambda) = -\frac{1}{48\pi^2} \frac{\rho}{\lambda^2} (\rho - \lambda)(\rho - 3\lambda).
\]

Note the \( \epsilon \) dependence cancels. We see that the \( \beta \)-function has zeros at \( \rho/\lambda = 0, 1, \) and \( 3 \). If it is the case that \( \lambda > 0 \), then the \( \beta \)-function is positive when \( \rho/\lambda < 0 \) and when \( 1 < \rho/\lambda < 3 \), and is negative when \( \rho/\lambda > 3 \) and \( 0 < \rho/\lambda < 1 \). Therefore, \( \beta_{\rho/\lambda} \) flows to \( \rho/\lambda = 1 \) and away from the other fixed points, meaning \( \rho/\lambda = 1 \) is an IR stable fixed point while \( \rho/\lambda = 0 \) or \( 3 \) are IR unstable, i.e. perturbing away from these fixed points we either flow to the stable fixed point of off to the strong coupling regime where perturbation theory breaks down. It is at the stable fixed point we see the emergent \( \mathcal{O}(2) \) symmetry.

c) Working in \( d = 4 \), i.e. in the \( \epsilon \rightarrow 0 \) limit, the \( \beta \)-functions

\[
\beta_\lambda = \frac{3}{16\pi^2} \left( \lambda^2 + \frac{\rho^2}{9} \right), \quad \beta_\rho = \frac{1}{8\pi^2} \left( \lambda \rho + \frac{2\rho^2}{3} \right),
\]

are only zero when both of the couplings are zero, \( \lambda = 0, \rho = 0 \). This is a Gaussian fixed point and is the only one we see in \( d = 4 \).

If we instead take \( d = 4 - \epsilon \), where the \( \beta \)-functions are

\[
\beta_\lambda = \frac{3}{16\pi^2} \left( \lambda^2 + \frac{\rho^2}{9} \right) - \lambda \epsilon, \quad \beta_\rho = \frac{1}{8\pi^2} \left( \lambda \rho + \frac{2\rho^2}{3} \right) - \rho \epsilon,
\]

we find that they have zeros when

\[
\{ \rho = 0, \lambda = 0 \}, \quad \{ \rho = 0, \lambda = \frac{16\pi^2}{3} \epsilon \}, \quad \{ \rho = 8\pi^2 \epsilon, \lambda = \frac{8\pi^2}{3} \epsilon \}, \quad \{ \rho = \frac{24\pi^2}{5} \epsilon, \lambda = \frac{24\pi^2}{5} \epsilon \}.
\]
The first fixed point is the Gaussian fixed point above, but we see that we now have three new fixed points emerging at finite $\epsilon$! These are Wilson-Fisher fixed points.

We can plot the $\beta$-functions as a vector flow on the space of couplings in $d = 4 - \epsilon$. Both from the plot and from the sign of the $\beta$-functions near the fixed points we see that the Gaussian fixed point is IR unstable and has two relevant directions. The second two fixed points have one relevant and one irrelevant direction, and the last fixed point is IR stable and has two irrelevant directions. Note that taking $\epsilon \to 0$, all the fixed points coalesce back into the single Gaussian fixed point.

The $O(2)$ symmetry is an asymptotic symmetry in the IR, emerging at the stable fixed point. The Wilson-Fisher fixed point describes a continuous phase transition between ordered and disordered phases, where we can calculate critical exponents as an expansion in $\epsilon$ for different systems in the universality class of this fixed point.

**Scale-invariant Quantum Mechanics**

We want to study a simple scale invariant quantum mechanical theory which exhibits interesting features that appear in strongly-interacting QFTs,

$$S[x] = \int dt \left( \frac{1}{2} (\partial_t \vec{x})^2 + g_0 \delta^{(2)}(\vec{x}) \right),$$

where the coordinates are $\vec{x} = (x, y)$ and the potential is $V(\vec{x}) = -g_0 \delta(\vec{x})$. Classically, at the level of the action, the theory is scale invariant as $[g_0] = 0$.

a) As the Hamiltonian of the model in position basis is

$$H = -\frac{1}{2} (\partial_x^2 + \partial_y^2) - g_0 \delta^{(2)}(\vec{x}),$$

the Schrödinger equation can be written as

$$\left( \frac{1}{2} \nabla^2 + E \right) \psi(x) = -g_0 \delta^{(2)}(\vec{x}) \psi(x) \rightarrow \left( \frac{p^2}{2} - E \right) \tilde{\psi}(p) = g_0 \int \frac{d^2 p'}{(2\pi)^2} \tilde{\psi}(p')$$

One solution is that of a free theory, where

$$\left( \frac{p^2}{2} - E \right) \tilde{\psi}(p) = 0,$$

with the condition that $\int d^2 p \tilde{\psi}(p) = 0$, equivalently $\psi(0) = 0$, and the solutions are plane waves which vanish at the origin and don’t care about the $\delta$-function potential.

The second solution is $\psi(0) = c$, a constant, and

$$\tilde{\psi}(p) = \frac{g_0}{p^2/2 - E} c.$$
This solution runs into problems if \( E > 0 \) as then there are momenta for which \( p^2 = 2E \) and the solution blows up. Thus we should consider boundstate solutions \( E = E_B < 0 \). We can insert the solution back into the Schrödinger equation and find a condition on the boundstate energy

\[
\tilde{\psi}(p) = \frac{g_0}{p^2/2 + E_B} \rightarrow \int \frac{d^2p}{(2\pi)^2} \frac{g_0}{p^2/2 + E_B} = 1.
\]

The integral diverges logarithmically, \( \int \frac{d^2p}{p^2} \rightarrow \infty \), but we can regulate the integral by imposing a cutoff at some large momentum \( \Lambda \), so that \( \int \frac{d^2p}{\Lambda^2} \sim \log \Lambda \). More explicitly,

\[
\int \frac{d^2p}{(2\pi)^2} \frac{g_0}{p^2/2 + E_B} = \frac{g_0}{2\pi} \log \left( 1 + \frac{\Lambda^2}{2E_B} \right) = 1
\]

is now the condition on the boundstate energy. Note that a solution only exists if \( g_0 > 1 \) meaning the potential we considered must be attractive. Solving for the boundstate energy we find

\[
E_B = \frac{\Lambda^2}{2} \frac{1}{e^{2\pi/g_0} - 1}, \tag{3}
\]

and we see what was happening before, removing the cutoff \( \Lambda \rightarrow \infty \), the bound state energy diverges \( E_B \rightarrow \infty \).

b) Given some cutoff \( \Lambda \), we can solve for \( g_0(\Lambda) \)

\[
g_0(\Lambda) = \frac{2\pi}{\log \left( 1 + \frac{\Lambda^2}{2E_B} \right)},
\]

where we are allowing the bare coupling to be cutoff dependent. Instead of a dimensionless coupling \( g_0 \), we now have a theory with a dimensionful scale, \( E_B \). This is dimensional transmutation. We had an effective coupling and specified a scale and a value of \( g_0 \) measured at that scale, but now we see that the scale was all we needed. The cutoff will not appear in the observables, which will actually depend on \( E_B \).\(^1\) Furthermore, as we take \( \Lambda \rightarrow \infty \), the bare coupling vanishes \( g_0(\Lambda) \rightarrow 0 \) and the theory becomes free, this is asymptotic freedom.\(^2\)

We also note that for small \( g_0 \) in Eq. (3), i.e. in the perturbative regime, we cannot perform a power series expansion of the dimensionful parameter \( E_B \) in terms of the coupling \( g_0 \), and thus dimensional transmutation is inherently a nonperturbative phenomena (we won’t see it in perturbation theory in \( g_0 \)). We can compute the \( \beta \)-function

\[
\beta(g_0) = \Lambda \frac{dg_0(\Lambda)}{d\Lambda} = -\frac{g_0^2}{\pi} \left( 1 - e^{-2\pi/g_0} \right).
\]

and see that it can be expressed as a function of \( g_0 \) without explicit \( \Lambda \) dependence. The \( \beta \)-function has a fixed point at \( g_0 = 0 \) and as we approach the fixed point, i.e. at small \( g_0 \), the nonperturbative term is suppressed and we have \( \beta \approx -\frac{g_0^2}{\pi} \). So \( g_0 = 0 \) is an attractive fixed point.

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\(^1\)In QCD, we should expect the same thing, that an a scale \( \Lambda_{\text{QCD}} \) will enter into physical quantities, and the bound states of the theory (mesons, baryons, etc.) should have masses that are given by some constant times this one dimensionful scale \( \Lambda_{\text{QCD}} \).

\(^2\)again, just like in QCD.