# Supersymmetric Wilson loops in $\mathcal{N} = 4$ SYM and pure spinors

Anatoly Dymarsky

Institute for Advanced Study

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# Plan of the talk

- Motivation: why SUSY Wilon loops
  historical overview and the problem to solve
- Supersymmetry and pure spinors
  space of pure spinors and the corresponding loops
- Factorization over symmetry group $SO(5,1) \times SO(6)$
  explicit parametrization of moduli space
- Future directions

# Motivation

Motivation: supersymmetric observables in $\mathcal{N} = 4$ SYM

- check of AdS/CFT correspondence
- BPS sector of $\mathcal{N} = 4$ SYM and integrability
  explicit answers provide insight about dynamics

## Problem

- Describe all supersymmetric Wilson loops

# SUSY Wislon loops: historical overview

## SUSY loops on $R^4$

Zarembo

$$W = \exp \int_\gamma dx^\mu (A_\mu - i\Theta_{\mu\nu}\Phi_\nu)$$

$$\gamma \in R^4$$

$$\Theta_{\mu\nu} = \delta_{\mu\nu}$$

## SUSY loops on $S^3/R^3$

Drukker, Giombi, Ricci, Trancanelli

$$W = \exp \int_\gamma dx^i (A_i - i\Theta_{ij}\Phi_j)$$

$$\gamma \in R^3$$

$$\Theta^{ij} = \frac{\delta^{ij}(1-\mathrm{x}^2)+2\mathrm{x}^j\mathrm{x}^j-2\epsilon^{ijk}\mathrm{x}^k}{1+\mathrm{x}^2}$$

# SUSY Wislon loops in bulk: historical overview

## Dual strings are pseudo-holomorphic surfaces

A.D., Gubser, Guralnik, Maldacena; Drukker, Giombi, Ricci, Trancanelli

- one can define pseudo-complex structure $J$ in the bulk

$$(\delta^M_N + iJ^M_N)\bar{\partial}X^N = 0$$

as a result the action is calibrated by $\int P[J]$

supersymmetry follows from $\partial X^N(\delta^M_N + iJ^M_N)\Gamma_M \varepsilon = 0$

# Most general SUSY loop

$$W = \exp \int_\gamma ds (A_\mu \dot{x}^\mu - i\Theta^i \Phi_i) = \exp \int_\gamma ds \; v^M A_M$$

What are the contours $\gamma$ and couplings $\Theta^i$ that make $W$ SUSY?

A.D., Pestun, to be published

.. and how to find $J$ in a general case?

- SUSY transformation $Q_\epsilon$ acts as follows

$$\delta A_M = \varepsilon \Gamma_M \Psi$$

$$v^M \Gamma_M \varepsilon = 0$$

$$v_M = \{\dot{x}_\mu, -i\Theta_i\}$$

# Local SUSY condition

For the SUSY generator $Q_\varepsilon$ find all *couplings* $v_M = \{\dot{x}_\mu, -i\Theta_I\}$ such that

$$v^M \Gamma_M \varepsilon(x) = 0$$

Let's define vector $u^M \equiv \varepsilon \Gamma^M \varepsilon$

Locally the solution is

- if $u^M \neq 0$ $v^M = u^M$ is a unique solution
- if $u^M = 0$ the spinor $\varepsilon$ is *pure*
  pure spinor is annihilated by half of gamma-matrixes
  $v^M \in V^{0,1} \cong \mathbb{C}^5$

# Overview of pure spinors in $\mathbb{R}^{2n}$

Pure spinor $\varepsilon$ is defined through the condition that it is annihilated by half of gamma algebra

- pure spinor defines (almost) complex structure $J^M_N$ in $\mathbb{R}^{2n}$ through

$$(\delta^M_N + iJ^M_N)\Gamma_M \varepsilon = 0$$

- pure spinor is a vacuum in the Fock representation. Creating $\Gamma$ and annihilating $\bar{\Gamma}$ operators obey the algebra of external forms

$$\{\Gamma, \Gamma\} = \{\bar{\Gamma}, \bar{\Gamma}\} = 0 \quad \{\Gamma_I, \bar{\Gamma}_{\bar{J}}\} = \delta_{I\bar{J}}$$

Pure spinor is the choice of the complex structure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Klifford algebra can be represented through the algebra of the external forms

$$\Gamma_{\bar{I}_1}..\Gamma_{\bar{I}_k}\varepsilon \leftrightarrow dz_{\bar{I}_1} \wedge .. \wedge dz_{\bar{I}_k}$$

$$\Gamma_{\bar{I}} \leftrightarrow dz_{\bar{I}}$$ and $$\Gamma_I \leftrightarrow i_{z_I}$$

# General SUSY Wilson loop

The SUSY generator $Q_\varepsilon$ specifies the conformal Killing spinor

$$\varepsilon(x) = \varepsilon_S + x^\mu \Gamma_\mu \varepsilon_C$$

At each point $x$ we define $$u^M(x) = \varepsilon \Gamma^M \varepsilon$$

- if $u^\mu \neq 0$ locally there is unique SUSY Wilon loop with $\dot{x}^\mu = u^\mu$ and $-i\Theta^i = u^i$
  $u^\mu$ is a vector field of the infinitesimal conformal transformation generated by $Q_\varepsilon^2$
- if $u^\mu = 0$ but $u^M \neq 0$ locally the only SUSY Wilson loop is a local operator $u^i \Phi_i$
- if $u^M = 0$ locally there is five-dimensional space of (antiholomorphic) couplings $v^M = \{\dot{x}^\mu, -i\Theta^i\}$ that define SUSY Wilson loop
  if $\varepsilon(x)$ is pure we have rich variety of choices for $\gamma$ and $\Theta$

# When $\varepsilon(x) = \varepsilon_S + x^M \Gamma_M \varepsilon_C$ is pure?

When (for which $\varepsilon_S, \varepsilon_C$) $\varepsilon(x)$ is pure, and where?

equation to solve $\varepsilon(x)\Gamma^M \varepsilon = 0$

- let's assume $\varepsilon(x)$ is pure somewhere (at the origin); then $\varepsilon_S$ is pure and defines complex structure on $\mathbb{R}^{10}$

let's represent $\varepsilon(x)$ using form notations

- $\varepsilon_C = (\mathtt{v} + \mathtt{m} + \mathtt{w})\epsilon_S$ where $\mathtt{v}, \mathtt{m}, \mathtt{w}$ are antiholomorphic 1-form, 3-form and 5-form in $\mathbb{C}^5$ correspondingly
- $\varepsilon(x) = ((1 + 2i_x \mathtt{v}) + (\xi \wedge \mathtt{v} + 2i_x \mathtt{m}) + (\xi \wedge \mathtt{m} + 2i_x \mathtt{w}))\varepsilon_S$ where $\xi_{\bar{I}} = \frac{1}{2} g_{\bar{I}J} x^J$

$\varepsilon(x)$ is pure where $(\xi + x^2 \mathtt{v}) \wedge \mathtt{m} = 0$ $\mathtt{w} = 0$ and $\mathtt{m}$ is decomposable

# Pure spinor in the bulk $AdS_5 \times S^5 \cong \mathbb{R}^{10}$

spinor is pure on submanifold $\Sigma_{\mathbb{C}} \in \mathbb{R}^{10}$ defined through

$$(\xi + x^2 \mathtt{v}) \wedge \mathtt{m} = 0$$

and $\mathtt{m}$ is decomposable

- if $\mathtt{m} = 0$ the spinor is pure everywhere in the bulk $\mathbb{R}^{10}$
- if $\mathtt{m} \neq 0$ the spinor is pure on $\Sigma_{\mathbb{C}} = \mathbb{R}^6 / S^6 \in \mathbb{R}^{10}$

$\varepsilon$ is pure on $\Sigma_{\mathbb{C}}$ hence it defines almost complex structure $J$ through

$$(\delta^M_N + i J^M_N)\Gamma_M \varepsilon(x) = 0$$

as a result any pseudo-holomorphic surface in the bulk is necessarily supersymmetric

# Back to $\mathcal{N} = 4$ SYM

On the boundary the spinor $\varepsilon(x)$ is pure on $\Sigma = \Sigma_{\mathbb{C}} \bigcap \mathbb{R}^4_{\rm spt} \in \mathbb{R}^{10}$

- if $\mathtt{m} = 0$ the spinor is pure everywhere on $\Sigma = \mathbb{R}^4$
- if $\mathtt{m} \neq 0$ the spinor is pure on $\Sigma = \mathbb{R}^6 \bigcap \mathbb{R}^4 = \mathbb{R}^{1,2,3,4}$

if $\mathtt{m} = 0$ the remaining parameters are

- pure spinor $\varepsilon_S$ i.e. choice of complex structure $J_N^M$ in $\mathbb{R}^{10}$
- vector $\mathtt{v}$

in the simplest case

- $\varepsilon_S$ is defines through

$x^I = x^{2I-1} - ix^{2I}$ for $I = 1..4$ and $x^{I=5} = x^9 + ix^{10}$

- $\mathtt{v}^M = 0$

the corresponding SUSY Wilson loops are those introduced by Zarembo with $\Theta_{\mu\nu} = \delta_{\mu\nu}$

# General $J_N^M$ i.e. general pure spinor $\varepsilon_S$

most general complex structure $J_N^M$ could be brought to the form

$$x^{I=1} = \cos\frac{\alpha}{2}x_1 - \sin\frac{\alpha}{2}x_6 + i\left(\sin\frac{\alpha}{2}x_2 - \cos\frac{\alpha}{2}x_5\right)$$
$$x^{I=2} = \cos\frac{\alpha}{2}x_2 + \sin\frac{\alpha}{2}x_5 - i\left(\sin\frac{\alpha}{2}x_1 + \cos\frac{\alpha}{2}x_6\right)$$
$$x^{I=3} = \cos\frac{\beta}{2}x_3 - \sin\frac{\beta}{2}x_8 + i\left(\sin\frac{\beta}{2}x_4 - \cos\frac{\beta}{2}x_7\right)$$
$$x^{I=4} = \cos\frac{\beta}{2}x_4 + \sin\frac{\beta}{2}x_7 - i\left(\sin\frac{\beta}{2}x_3 + \cos\frac{\beta}{2}x_8\right)$$
$$x^{I=5} = x_9 + ix_{10}$$

if $\mathrm{v}^M = 0$ the Wilosn loop is

$$\exp\int_\gamma dx^1\left(A_1 - i\left(\cos^{-1}\alpha\Phi_1 - i\tan\alpha\Phi_2\right)\right) + dx^2\left(A_2 - i\left(\cos^{-1}\alpha\Phi_2 + i\tan\alpha\Phi_1\right)\right) + \ldots + ds\left(\Phi_5 + i\Phi_6\right) \tag{1}$$

# What Wilson loops are the same?

Naively we have 12 parameters $\alpha, \beta, \mathtt{v}^M$ to label different $Q_\varepsilon$ and the corresponding SUSY Wilson loops. But are they all different?

Clearly not! Not all symmetries were use up

- Action of the symmetry group $SO(5,1) \times SO(6)$ can be embedded into the action of the Klifford algebra on $\mathbb{R}^{11,1}$ on the $64$ components spinor

$$\varepsilon_{64}(\alpha, \beta, \mathtt{v}) = \begin{pmatrix} \varepsilon_S \\ 0 \\ 0 \\ \varepsilon_C \end{pmatrix}$$

# $SO(5,1) \times SO(6)$ invariants

We can form various $SO(5,1) \times SO(6)$ invariants first by defining a tensor in $\mathbb{R}^{5,1} \times \mathbb{R}^6 = \mathbb{R}^{11,1}$

$$\bar{\varepsilon}_{64}\gamma_{i_1}..\gamma_{i_p}\gamma_{j_1}..\gamma_{j_q}\varepsilon_{64}$$

and then contracting the indexes

instead of $12$ there are just two invariants in the case when $\mathtt{m} = 0$:

$\beta$ and $\mathtt{v}_5/\mathtt{v}_9$ while $\alpha$ and all other $\mathtt{v}_M$ are zero

# Cases when $\mathtt{m} \neq 0$

Dilatation and the $SO(4) \times SO(6)$ rotation can be used to bring $\mathtt{m}$ to the canonical form $\mathtt{m} = dz_1 \wedge dz_2 \wedge dz_3$

- $\Sigma = \mathbb{R}^1$ there are two parameters $\mathtt{v}_5, \mathtt{v}_7$

$$W[\varphi_1, \varphi_2] = \exp \int ds \left( A_1 \; \Phi_3 \; \Phi_6 \; \Phi_1 \; \Phi_5 \; \Phi_4 \right) \begin{pmatrix} \mathbb{I}_{3\times 3} \\ -i\Theta \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$\Theta = \begin{pmatrix} \frac{1-2i\mathtt{v}_5 x_1 + \left(1-\mathtt{v}_5^2-\mathtt{v}_7^2\right)x_1^2}{1-2i\mathtt{v}_5 x_1 + \left(1-\mathtt{v}_5^2+\mathtt{v}_7^2\right)x_1^2} \\ \frac{2\mathtt{v}_7 x_1 (1-i\mathtt{v}_5 x_1)}{1-2i\mathtt{v}_5 x_1 + \left(1-\mathtt{v}_5^2+\mathtt{v}_7^2\right)x_1^2} \\ \frac{2\mathtt{v}_7 x_1^2}{1-2i\mathtt{v}_5 x_1 + \left(1-\mathtt{v}_5^2+\mathtt{v}_7^2\right)x_1^2} \end{pmatrix}$$

# Cases when $\mathtt{m} \neq 0$, continuation

- $\Sigma = \mathbb{R}^2$ there are four parameters $\alpha, \beta, \mathrm{v}_9, \mathrm{v}_{10}$
- $\Sigma = \mathbb{R}^3$ there are two parameters $\alpha, \mathrm{v}_7$
- $\Sigma = \mathbb{R}^4$ there is only one non-trivial parameter $\mathrm{v}_5$

in the case $\Sigma = \mathbb{R}^4$ at the origin the contour $\gamma$ lies in the $1-2$ plane

$$dx^1(A_1 - i\Phi_1) + dx^2(A_2 - i\Phi_2) + ds\varphi(A_3 - iA_4)$$

at each point on $\mathbb{R}^4$ there is a plane of "allowed" directions for $\gamma$

- the "allowed" planes do not belong to the tangent bundle of a subsurface in $\mathbb{R}^4$
- "allowed" contours $\gamma$ may not be closed

# Outline

- SUSY Wilson loops in $\mathcal{N} = 2, 1$ SYM, in progress
- dynamics: *vev*'s and correlation functions in field theory (localization) and in the bulk
- surface operators