Stability, Topology, Holography: The many facets of quantum error correction
## Frontiers of Physics

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Quantum Supremacy!
A quantum computer can simulate efficiently any physical process that occurs in Nature.

(Maybe. We don’t actually know for sure.)
Decoherence

\[ \frac{1}{\sqrt{2}} \left( \begin{array}{c} \text{Environment} \\ \end{array} \right) \]
Decoherence explains why quantum phenomena, though observable in the microscopic systems studied in the physics lab, are not manifest in the macroscopic physical systems that we encounter in our ordinary experience.
To resist decoherence, we must prevent the environment from “learning” about the state of the quantum computer during the computation.
Quantum error correction

The protected “logical” quantum information is encoded in a highly entangled state of many physical qubits.

The environment can't access this information if it interacts locally with the protected system.
The thermal time scale thus sets a (weak) limit on the length of time that a quantum calculation can take.
“...small errors will accumulate and cause the computation to go off track.”
Therefore we think it fair to say that, unless some unforeseen new physics is discovered, the implementation of error-correcting codes will become exceedingly difficult as soon as one has to deal with more than a few gates. In this sense the large-scale quantum machine, though it may be the computer scientist's dream, is the experimenter's nightmare.
Combining repetition codes for bit flips and phase errors (Shor code).

A quantum version of the classical Hamming code (Steane code).
Entanglement purification and teleportation for faithful transmission of quantum information through noisy channels.
Calderbank-Shor-Steane (CSS) Codes: the first family of good quantum codes.
Quantum stabilizer codes: the quantum analogue of additive classical codes.
Fault-tolerant syndrome measurement, using encoded ancillas, verified offline.

Universal gates acting on encoded quantum data, using “magic states” verified offline.
Threshold Accuracy for Quantum Computation

E. Knill, R. Laflamme, W. Zurek

(Submitted on 8 Oct 1996 (v1), last revised 15 Oct 1996 (this version, v3))

We have previously (quant-ph/9608012) shown that for quantum memories and quantum communication, a state can be transmitted over arbitrary distances with error $\epsilon$ provided each gate has error at most $c\epsilon$. We discuss a similar concatenation technique which can be used with fault tolerant networks to achieve any desired accuracy when computing with classical initial states, provided a minimum gate accuracy can be achieved. The technique works under realistic assumptions on operational errors. These assumptions are more general than the stochastic error heuristic used in other work. Methods are proposed to account for leakage errors, a problem not previously recognized.

Fault Tolerant Quantum Computation with Constant Error

Dorit Aharonov (Physics and computer science, Hebrew Univ.), Michael Ben-Or (Computer science, Hebrew univ.)

(Submitted on 14 Nov 1996 (v1), last revised 15 Nov 1996 (this version, v2))

Recently Shor showed how to perform fault tolerant quantum computation when the error probability is logarithmically small. We improve this bound and describe fault tolerant quantum computation when the error probability is smaller than some constant threshold. The cost is poly/logarithmic in time and space, and no measurements are used during the quantum computation. The result holds also for quantum circuits which operate on nearest neighbors only. To achieve this noise resistance, we use concatenated quantum error correcting codes. The scheme presented is general, and works with all quantum codes that satisfy some restrictions, namely that the code is "proper".
Scalable quantum computing

**Quantum Accuracy Threshold Theorem**: Consider a quantum computer subject to quasi-independent noise with strength $\varepsilon$. There exists a constant $\varepsilon_0 > 0$ such that for a fixed $\varepsilon < \varepsilon_0$ and fixed $\delta > 0$, any circuit of size $L$ can be simulated by a circuit of size $L^*$ with accuracy greater than $1 - \delta$, where, for some constant $c$,

$$L^* = O\left[ L \left( \log L \right)^c \right]$$

assuming:

- parallelism, fresh qubits (*necessary* assumptions)
- nonlocal gates, fast measurements, fast and accurate classical processing, no leakage (*convenient* assumptions).

“Practical” considerations:
Resource requirements, systems engineering issues

Matters of “principle”:
Conditions on the noise model, what schemes are scalable, etc.

Aharonov, Ben-Or
Kitaev
Laflamme, Knill, Zurek
Aliferis, Gottesman, Preskill
Reichardt
Limitations on transversal (local unitary) logical gates

The logical gates close to the identity that can be executed with local unitary transformations form a (perhaps trivial) Lie algebra.

\[ U = I + \varepsilon(A_1 + A_2 + \cdots + A_n) \]

If the code can “detect” a weight-one error, then for each \( i \):

\[ \Pi A_i \Pi \propto \Pi \]

(\( \Pi \) = projector onto code space)

If \( U \) preserves the code space, then \( U \) acts trivially on code space:

\[ U \Pi = \Pi U \Pi = \Pi \]

There are no transversal gates close to the identity. The group generated by transversal gates is finite and hence nonuniversal. Which logical gates can be executed transversally depends on what code we use.

Eastin-Knill 2009
Evading the Eastin-Knill Theorem


Multiple partitions. Jochym-O’Connor and Laflamme (2013). One set of operations transversal with respect to one partition and another set with respect to another. \[ \langle \mathcal{L}_1, \mathcal{L}_2 \rangle = \text{all} \]

Code switching. Duclos-Cianci and Poulin (2014). \[ C_1 \leftrightarrow C_2 \]
For example, fix the gauge of a subsystem code in two distinct ways. Need to switch fault-tolerantly.

Code drift. Paetznick and Reichardt (2013). \[ \mathcal{L} : C_1 \rightarrow C_2 \]
Need to return to the original code fault-tolerantly.
Alexei Kitaev
9 April 1997 ... An exciting day!
Nonabelian anyons

Quantum information can be stored in the collective state of exotic particles in two spatial dimensions ("anyons").

The information can be processed by exchanging the positions of the anyons (even though the anyons never come close to one another).
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The information can be processed by exchanging the positions of the anyons (even though the anyons never come close to one another).
Topological quantum computation

(Kitaev ‘97, FLW ‘00)

create pairs

braid

braid

braid

annihilate pairs?

create pairs

Kitaev

Freedman
Topological quantum computation (Kitaev '97, FLW '00)

annihilate pairs?

braid

braid

braid

create pairs

Kitaev

Freedman
Topological quantum computation

The computation is intrinsically resistant to decoherence. If the paths followed by the particles in spacetime execute the right braid, then the quantum computation is guaranteed to give the right answer!
Kitaev’s (2001) magic trick: sawing an *electron* in half!
Majorana fermion

topological superconductor

Majorana fermion

add an electron

conventional superconductor
Majorana fermion | topological superconductor | Majorana fermion

conventional superconductor | topological superconductor | conventional superconductor
Majorana fermion \[\downarrow\]

*topological superconductor*

Majorana fermion

conventional superconductor

Kouwenhoven


Albrecht, Higginbotham, Madsen, Kuemmeth, Jespersen, Nygard, Krogstrup, Marcus (2016).

Marcus
Daniel Gottesman,
Fault tolerance in small experiments

David DiVincenzo,
Quantum error correction and the future of solid state quantum computing

Philipp Schindler,
Quantum error correction with trapped ions

Michel Devoret,
Quantum error correction in superconducting circuits
Topological quantum error correction

Classical memory $\approx$ ferromagnet order
Robust bit

Quantum memory $\approx$ topological order
Robust qubit
Red or green (abelian) anyons
Realize physically, or simulate with generic hardware.

The best way to reduce the overhead cost of fault-tolerance: better gates!

Gradually the distinction between error correction in “hardware” and “software” will blur. We will learn to make better gates in many platforms by incorporating error correction at the physical level.

For example:

“0-Pi qubit” – robust degeneracy in a superconducting circuit:

\[ E \approx f(2\theta) + O\left(\exp(-c(size))\right) \]

Self-correcting quantum memory

1) Finite-dimensional spins.

2) Bounded-strength local interactions.

3) Nontrivial codespace.

4) Perturbative stability.

5) Efficient decoding.

6) Memory time exponential in system size at nonzero temperature.

The 4D toric code obeys all the rules, but what about < 4 dimensions?

Dennis, Landahl, Kitaev, Preskill (2002).
A local process starting from the “vacuum” (no excitations) and arriving at a state where a single topological defect is isolated from all others by distance at least $R$, must pass through a state whose “energy” is logarithmic in $R$.

This energy barrier impedes thermal defect diffusion, enhancing the stability of the quantum memory.
hep-th papers with “entanglement” in the title
Two amazing ideas:

- Holographic correspondence
- Quantum error correction

Are they closely related?

-- Scrambled encoding on boundary, protected against erasure.
-- Entanglement seems to be the glue holding space together.
-- Illustrates the surprising unity of physics.
-- Toward accessible experiments probing quantum gravity?


Building on Almheiri, Dong, Harlow (2014).
PARADOX!
When the theories we use to describe Nature lead to unacceptable or self-contradictory conclusions, we are faced with a great challenges and great opportunities…

Planck 1900

“The ultraviolet catastrophe”
In thermal equilibrium at nonzero temperature, the electromagnetic field carries an infinite energy per unit volume …

The end of classical physics!

Hawking 1975

“The information loss puzzle”
The radiation emitted by an evaporating black hole is featureless, revealing nothing about how the black hole formed …

The end of quantum physics? (Or of relativistic causality?)
A black hole in a bottle

We can describe the formation and evaporation of a black hole using an “ordinary” quantum theory on the walls of the bottle, where information has nowhere to hide (Maldacena 1997).


So at least in the one case where we think we understand how quantum gravity works, a black hole seems not to destroy information!

Even so, the mechanism by which information can escape from behind a putative event horizon remains murky.

Indeed, it is not clear whether or how the boundary theory describes the experience of observers who cross into the black hole interior, or even if there is an interior!
Bulk/boundary duality: an exact correspondence

-- Weakly-coupled gravity in the bulk $\leftrightarrow$ strongly-coupled conformal field theory on boundary.

-- Complex dictionary maps bulk operators to boundary operators.

-- Emergent radial dimension can be regarded as an RG scale.

-- Semiclassical (sub-AdS scale) bulk locality is highly nontrivial.

-- Geometry in the bulk theory is related to entanglement structure of the boundary theory.
Holographic entanglement entropy

To compute entropy of region $A$ in the boundary field theory, find minimal area of the bulk surface $\gamma_A$ with the same boundary \((\text{Ryu-Takayanagi 2006})\).

$$S(A) = \frac{1}{4G_N} \text{Area}(\gamma_A) + \cdots$$
Perfect tensors

The tensor $T$ arises in the expansion of a pure state of $2n$ $v$-dimensional “spins” in an orthonormal basis.

$$|\psi\rangle = \sum_{a_1,a_2,\ldots,a_{2n}} T_{a_1a_2\ldots a_{2n}} |a_1a_2\ldots a_{2n}\rangle$$

$T$ is perfect if the state is maximally entangled across any cut, i.e. for any partition of the $2n$ spins into two sets of $n$ spins. (State is absolutely maximally entangled.)

By transforming kets to bras, $T$ also defines $3 \rightarrow 3$ unitary, $2 \rightarrow 4$ and $1 \rightarrow 5$ isometries.

These are the isometric encoding maps (up to normalization) of quantum error-correcting codes. The $2 \rightarrow 4$ map encodes two qubits in a block of 4, and corrects 1 erasure. The $1 \rightarrow 5$ map encodes one qubit in a block of 5, and corrects 2 erasures.
Erasure correction

The 1→5 isometric map encodes one qubit in a block of 5, and corrects two erasures.

$$\sum_{a_1 \ldots a_6} T_{a_1 \ldots a_6} |a_2 a_3 a_4 a_5 a_6 \rangle \langle a_1 |$$

Consider maximally entangling a *reference qubit* $R$ with the encoded qubit. Suppose two physical qubits (the subsystem $E$) are removed, while their complement $E^c$ is retained.

Because the tensor $T$ is perfect, $RE$ is maximally entangled with $E^c$, hence $R$ is maximally entangled with a subsystem of $E^c$. Thus the logical qubit can be decoded by applying a unitary decoding map to $E^c$ alone; $E$ is not needed.

Likewise, we may apply any logical operator to the encoded qubit by acting on $E^c$ alone. (The logical operation can be *cleaned* so it has no support on the erased qubits.)

We say qubits are erased if they are removed from the code block. But we know *which* qubits were erased and may use that information in recovering from the error.
Holographic quantum error-correcting codes are constructed by contracting perfect tensors according to a tiling of hyperbolic space by polygons.

The code is an isometric embedding of the bulk Hilbert space into the boundary Hilbert space, obtained by composing the isometries associated with each perfect tensor.
Ryu-Takayanagi Formula

Consider a holographic state $|\psi\rangle$ (no dangling bulk indices), and a geodesic cut $\gamma_A$ through the bulk with indices on the cut labeled by $i$. Indices of $A$ are labeled by $a$ and indices of $A^c$ labeled by $b$.

$$|\psi\rangle = \sum_{a,b,i} |a\rangle_A \otimes |b\rangle_{A^c} P_{ai} Q_{bi} = \sum_i |P_i\rangle_A \otimes |Q_i\rangle_{A^c}$$

For a holographic state on a tiling with nonpositive curvature, the tensors $P$ and $Q$ are both isometries, if $A$ is connected (max-flow min-cut argument).

If each internal index takes $v$ values, there are $v^{|\gamma|}$ terms in the sum over $i$. and the vectors $\{ |P_i\rangle \}$, $\{ |Q_i\rangle \}$ are orthonormal. Therefore

$$S(A) = |\gamma_A| \log v$$
Protection against erasure

For a connected region $A$ on the boundary there is a corresponding geodesic $\gamma_A$. Bulk operators in the wedge between $A$ and $\gamma_A$ can be reconstructed on the boundary in region $A$.

Operators deeper in the bulk have better protection against erasure on the boundary.

Bulk operators at the center of the bulk are robust against erasure of up to half of the boundary qubits.
Holographic black holes

-- Most boundary states correspond to large black holes in the bulk.

-- Bulk local operators acting outside the black hole can be reconstructed on the boundary.

-- Uncontracted bulk indices at the horizon, the black hole microstates, are also mapped to the boundary.

-- Encoding isometry becomes trivial as black hole grows to fill the whole bulk.
Holographic quantum codes

-- Nicely capture some central features of full blown gauge/gravity duality, and provide an explicit dictionary relating bulk and boundary observables.

-- Realize exactly the Ryu-Takayanagi relation between boundary entanglement and bulk geometry (with small corrections in some cases).

-- But … so far these models are not dynamical, and do not address bulk locality at sub-AdS distance scales.
Quantumists ≈ Biologists

quantum gravity = life
boundary theory = chemistry
quantum information theorists = chemists
quantum gravity theorists = biologists

what we want = molecular biology
black hole information problem = fruit fly
understanding the big bang = curing cancer

Slide concept stolen from Juan Maldacena
Ooguri: I see that this new joint activity between quantum gravity and quantum information theory has become very exciting. Clearly entanglement must have something to say about the emergence of spacetime in this context.

Witten: I hope so. I’m afraid it’s hard to work on, so in fact I’ve worked with more familiar kinds of questions.
“Now is the time for quantum information scientists to jump into .. black holes”

Beni Yoshida
QuantumFrontiers.com
March 2015
Deep insights into the quantum structure of spacetime will arise from laboratory experiments studying highly entangled quantum systems.
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