Ph 219/CS 219

Problem Set 2 Solutions

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2.1 Which state did Alice make?

Bob is trying to determine which of two possible states (ρ_1 or ρ_2) that Alice has prepared. Bob also knows the probabilities of either state being prepared (namely, p_1 and $p_2 = 1 - p_1$). Given that Bob performs a 2-outcome POVM, we will calculate his best strategy.

a) Bob's possible outcomes are nonnegative Hermitian operators E_1 and $E_2 = I - E_1$. If Alice prepares ρ_1 (with probability p_1), then Bob will guess incorrectly if his measurement projects onto E_2 instead of E_1. Likewise, Bob's guess will be wrong whenever his measurement of ρ_2 projects onto E_1 instead of E_2. The probability of error for this strategy is then:

$$\begin{aligned}
p_{\text{error}} &= \text{Prob}\left[\text{Alice prepares } \rho_1 \text{ and Bob guesses } \rho_2\right] \; + \\
& \text{Prob}\left[\text{Alice prepares } \rho_2 \text{ and Bob guesses } \rho_1\right] \\
p_{\text{error}} &= p_1 \ \text{tr}\left(\rho_1 E_2\right) + p_2 \ \text{tr}\left(\rho_2 E_1\right) \\
&= p_1 \ \text{tr}\left(\rho_1 (I - E_1)\right) + p_2 \ \text{tr}\left(\rho_2 E_1\right) \\
&= p_1 \ \text{tr}\left(\rho_1\right) - p_1 \ \text{tr}\left(\rho_1 E_1\right) + p_2 \text{tr}\left(\rho_2 E_1\right) \\
&= p_1 \ \text{tr}\left(\rho_1\right) + \text{tr}\left(-p_1 \rho_1 E_1 + p_2 \rho_2 E_1\right) \\
p_{\text{error}} &= p_1 \ \text{tr}\left(\rho_1\right) + \text{tr}\left((p_2 \rho_2 - p_1 \rho_1) E_1\right)
\end{aligned}$$

The complete trace of a density operator is one, so the left term reduces to p_1. We can diagonalize the Hermitian operator $p_2\rho_2 - p_1\rho_1 = \sum_i |i\rangle\lambda_i\langle i|$ and evaluate the trace in this diagonal basis:

$$\begin{aligned}
p_{\text{error}} &= p_1 + \sum_j \langle j| \left(\sum_i |i\rangle \lambda_i \langle i| E_1 \right) |j\rangle \\
&= p_1 + \sum_j \sum_i \langle j|i\rangle \lambda_i \langle i|E_1|j\rangle \\
&= p_1 + \sum_j \sum_i \delta_{ij} \lambda_i \langle i|E_1|j\rangle
\end{aligned}$$

$$p_{\text{error}} = p_1 + \sum_i \lambda_i \langle i|E_1|i\rangle$$

b) The quantity $\langle i|E_1|i\rangle$ is nonnegative because E_1 is given as a nonnegative operator. Similarly, $\langle i|E_2|i\rangle$ is nonnegative. Adding these together, we find:

$$\langle i|E_1|i\rangle + \langle i|E_2|i\rangle = \langle i|\left(E_1 + E_2\right)|i\rangle = \langle i|I|i\rangle = 1$$

Thus, $\langle i|E_1|i\rangle$ is bounded between zero and one. In order to minimize p_{error}, Bob wants to choose E_1 in such a way that $\langle i|E_1|i\rangle$ is maximized whenever λ_i is negative and $\langle i|E_1|i\rangle$ is minimized whenever λ_i is positive. (We can allow any value when $\lambda_i = 0$ because it won't contribute to the sum in p_{error}.)

Explicitly, let $(E_1)_{\text{optimal}} = \sum_{i:\lambda_i<0} |i\rangle\langle i|$. That is, E_1 is the projector onto the space of eigenstates of $p_2\rho_2 - p_1\rho_1$ with negative eigenvalues. Since the set of $\{|i\rangle\}$ eigenstates form an orthonormal basis of Alice's system, the positive and negative eigenstates are orthogonal. The projection of $(E_1)_{\text{optimal}}$ onto a positive eigenstate is then zero.

Note that the eigenvalues of $(E_1)_{\text{optimal}}$ are zeroes and ones, so it is indeed a nonnegative operator. Furthermore, it is Hermetian, and we can see that the resulting expression for E_2 is also a nonnegative Hermetian operator.

The probability of error for this choice becomes:

$$\begin{aligned}
(p_{\text{error}})_{\text{optimal}} &= p_1 + \sum_i \lambda_i \langle i|\left(E_1\right)_{\text{optimal}}|i\rangle \\
&= p_1 + \sum_{i:\lambda_i<0} \lambda_i \cdot 1 + \sum_{i:\lambda_i\geq 0} \lambda_i \cdot 0 \\
(p_{\text{error}})_{\text{optimal}} &= p_1 + \sum_{i:\lambda_i<0} \lambda_i
\end{aligned}$$

c) We can evaluate

$$\begin{aligned}
\text{tr}\,(p_2\rho_2 - p_1\rho_1) &= p_2\ \text{tr}\,\rho_2 - p_1\ \text{tr}\,\rho_1 \\
&= p_2 - p_1 \\
&= (1 - p_1) - p_1 \\
\text{tr}\,(p_2\rho_2 - p_1\rho_1) &= 1 - 2p_1\,.
\end{aligned}$$

Then $p_1 = \frac{1}{2}\{1 - \text{tr}\ (p_2\rho_2 - p_1\rho_1)\}$.

Alternatively, we can express the trace in terms of eigenvalues:

$$\text{tr}\ (p_2\rho_2 - p_1\rho_1) = \sum_i \lambda_i = \sum_{\text{pos}} \lambda_i + \sum_{\text{neg}} \lambda_i$$

where "pos" and "neg" refer to summing over only the positive or the negative eigenvalues.

The trace norm of the same operator has a similar form:

$$\|p_2\rho_2 - p_1\rho_1\|_{\text{tr}} = \text{tr}\ |p_2\rho_2 - p_1\rho_1| = \sum_i |\lambda_i| = \sum_{\text{pos}} \lambda_i - \sum_{\text{neg}} \lambda_i$$

The difference of these two traces yields:

$$\begin{aligned}
\text{tr}\ (p_2\rho_2 - p_1\rho_1) - \text{tr}\ |p_2\rho_2 - p_1\rho_1| &= \left(\sum_{\text{pos}} \lambda_i + \sum_{\text{neg}} \lambda_i\right) \\
&\quad - \left(\sum_{\text{pos}} \lambda_i - \sum_{\text{neg}} \lambda_i\right) \\
\text{tr}\ (p_2\rho_2 - p_1\rho_1) - \text{tr}\ |p_2\rho_2 - p_1\rho_1| &= 2 \sum_{\text{neg}} \lambda_i
\end{aligned}$$

Then $\sum_{\text{neg}} \lambda_i = \frac{1}{2}\{\text{tr}\ (p_2\rho_2 - p_1\rho_1) - \|p_2\rho_2 - p_1\rho_1\|_{\text{tr}}\}$.

We can now rewrite the error probability found in part b:

$$\begin{aligned}
(p_{\text{error}})_{\text{optimal}} &= p_1 + \sum_{\text{neg}} \lambda_i \\
&= \frac{1}{2}\{1 - \text{tr}\ (p_2\rho_2 - p_1\rho_1)\} \\
&\quad + \frac{1}{2}\{\text{tr}\ (p_2\rho_2 - p_1\rho_1) - \|p_2\rho_2 - p_1\rho_1\|_{\text{tr}}\} \\
(p_{\text{error}})_{\text{optimal}} &= \frac{1}{2}\{1 - \|p_2\rho_2 - p_1\rho_1\|_{\text{tr}}\}
\end{aligned}$$

We can check that this expression makes sense in the limits of either identical or orthogonal states ρ_1 and ρ_2.

First, suppose $\rho_1 = \rho = \rho_2$. We could interpret this case as Alice choosing one of two distinct preparations of the same mixed state ρ. Bob can't distinguish at all between the two preparations, so he must simply guess the state without gathering any additional information.

If Bob guesses ρ_1 with probability a and ρ_2 with probability $1 - a$, then his error probability is $p_{\text{error}} = p_1 (1 - a) + p_2 a = p_1 + a (p_2 - p_1)$. This is minimized by $a = 0$ if $p_2 > p_1$ and by $a = 1$ if $p_2 < p_1$. Bob's optimal strategy is to just guess the state with larger probability. We then expect to find the optimal error probability to be $\min\{p_1, p_2\}$.

$$
\begin{aligned}
(p_{\text{error}})_{\text{optimal}} &= \frac{1}{2}\{1 - \|p_2\rho_2 - p_1\rho_1\|_{\text{tr}}\} \\
&= \frac{1}{2}\{1 - \|p_2\rho - p_1\rho\|_{\text{tr}}\} \\
(p_{\text{error}})_{\text{optimal}} &= \frac{1}{2}\{1 - |p_2 - p_1| \, \|\rho\|_{\text{tr}}\}
\end{aligned}
$$

Since ρ is a density operator, its eigenvalues are nonnegative real numbers that sum to one. Then $\|\rho\|_{\text{tr}} = \text{tr}\left(\rho^\dagger\rho\right)^{\frac{1}{2}} = \text{tr}\,|\rho| = \text{tr}\,\rho = 1$.
Let $p_{\max} = \max\{p_1, p_2\}$ and let $p_{\min} = \min\{p_1, p_2\}$.
Note that $p_{\max} + p_{\min} = 1$ and $p_{\max} - p_{\min} = |p_2 - p_1|$.

$$
\begin{aligned}
(p_{\text{error}})_{\text{optimal}} &= \frac{1}{2}\{1 - |p_2 - p_1| \, \|\rho\|_{\text{tr}}\} \\
&= \frac{1}{2}\{(p_{\max} + p_{\min}) - (p_{\max} - p_{\min}) \cdot 1\} \\
(p_{\text{error}})_{\text{optimal}} &= p_{\min}
\end{aligned}
$$

This matches the error probability of the optimal strategy determined above for the case of indistinguishable preparations.

Second, suppose that ρ_1 is orthogonal to ρ_2 (*i.e.*, $\rho_1\rho_2 = 0 = \rho_2\rho_1$, so their support is on orthogonal subspaces). For this case, we expect that Bob can perfectly distinguish between the two states, so the optimal error probability should be zero.

We can evaluate the trace norm:

$$
\|p_2\rho_2 - p_1\rho_1\|_{\text{tr}} = \text{tr}\left[(p_2\rho_2 - p_1\rho_1)^\dagger (p_2\rho_2 - p_1\rho_1)\right]^{\frac{1}{2}}
$$

$$
\begin{aligned}
&= \text{tr}\left[(p_2\rho_2)^2 - p_2p_1\rho_2\rho_1 - p_1p_2\rho_1\rho_2 + (p_1\rho_1)^2\right]^{\frac{1}{2}} \\
&= \text{tr}\left[(p_1\rho_1)^2 + (p_2\rho_2)^2\right]^{\frac{1}{2}} \\
&= \text{tr}\left[(p_1\rho_1)^2 + (p_2\rho_2)^2 + p_1p_2\rho_1\rho_2 + p_2p_1\rho_2\rho_1\right]^{\frac{1}{2}} \\
&= \text{tr}\left[(p_1\rho_1 + p_2\rho_2)^2\right]^{\frac{1}{2}} \\
\|p_2\rho_2 - p_1\rho_1\|_{\text{tr}} &= \text{tr}\,|p_1\rho_1 + p_2\rho_2|
\end{aligned}
$$

We can simultaneously diagonalize the orthogonal density operators $\rho_1 = \sum_a |a\rangle\lambda_a\langle a|$ and $\rho_2 = \sum_b |b\rangle\lambda'_b\langle b|$ such that $\langle a|b\rangle = 0\ \forall\ a, b$ whenever λ_a and λ'_b are both nonzero (in order to ensure $\rho_1\rho_2 = 0$). Additionally, we know that, when nonzero, the eigenvalues λ_a and λ'_b are always positive for density operators.

This indicates that the operator $p_1\rho_1 + p_2\rho_2$ has only nonnegative eigenvalues (namely, $\{p_1\lambda_a\} \cup \{p_2\lambda'_b\}$). We use this fact to calculate the optimal error probability:

$$
\begin{aligned}
(p_{\text{error}})_{\text{optimal}} &= \frac{1}{2}\{1 - \|p_2\rho_2 - p_1\rho_1\|_{\text{tr}}\} \\
&= \frac{1}{2}\{1 - \text{tr}\,|p_1\rho_1 + p_2\rho_2|\} \\
&= \frac{1}{2}\{1 - \text{tr}\,(p_1\rho_1 + p_2\rho_2)\} \\
&= \frac{1}{2}\{1 - p_1\ \text{tr}\,\rho_1 - p_2\ \text{tr}\,\rho_2\} \\
&= \frac{1}{2}\{1 - p_1 - p_2\} \\
(p_{\text{error}})_{\text{optimal}} &= 0
\end{aligned}
$$

Indeed, the expression for the optimal error probability becomes zero in the case of distinguishing between orthogonal states.

d) Now suppose that Alice decides at random (with $p_1 = p_2 = \frac{1}{2}$) to prepare one of two pure states $|\psi_1\rangle, |\psi_2\rangle$ of a single qubit, with

$$
|\langle\psi_1|\psi_2\rangle| = \sin(2\alpha)\ , \quad 0 \le \alpha \le \pi/4\ .
$$

With a suitable choice of basis, the two states can be expressed as

$$|\psi_1\rangle = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \qquad |\psi_2\rangle = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}.$$

To determine Bob's optimal two-outcome measurement, we begin by calculating:

$$\begin{aligned}
p_2\rho_2 - p_1\rho_1 &= \frac{1}{2}\{\rho_2 - \rho_1\} \\
&= \frac{1}{2}\{|\psi_2\rangle\langle\psi_2| - |\psi_1\rangle\langle\psi_1|\} \\
&= \frac{1}{2}\Bigg\{ \begin{pmatrix} \sin^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \cos^2\alpha \end{pmatrix} \\
&\quad - \begin{pmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{pmatrix}\Bigg\} \\
&= \frac{1}{2}\begin{pmatrix} \sin^2\alpha - \cos^2\alpha & 0 \\ 0 & \cos^2\alpha - \sin^2\alpha \end{pmatrix} \\
p_2\rho_2 - p_1\rho_1 &= \frac{1}{2}\begin{pmatrix} -\cos(2\alpha) & 0 \\ 0 & \cos(2\alpha) \end{pmatrix}
\end{aligned}$$

The eigenvalues of this operator are $\left\{+\frac{1}{2}\cos(2\alpha), -\frac{1}{2}\cos(2\alpha)\right\}$ with corresponding eigenstates $\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$. Note that $\cos(2\alpha) \geq 0$ for $0 \leq \alpha \leq \pi/4$.

From part a, the optimal choice for E_1 is to project onto the eigenstates of $p_2\rho_2 - p_1\rho_1$ with negative eigenvalues. For this problem, Bob should choose

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_2 = I - E_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We can use the expression from any of the previous parts to evaulte the error probability for the optimal two-outcome POVM:

$$\begin{aligned}
\text{(a)}\quad p_{\text{error}} &= p_1 + \sum_i \lambda_i \langle i|E_1|i\rangle \\
&= \frac{1}{2} + \left\{\frac{-\cos(2\alpha)}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}
\end{aligned}$$

$$
\begin{aligned}
& + \left\{ \frac{\cos(2\alpha)}{2} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \\
&= \frac{1}{2} - \frac{\cos(2\alpha)}{2} \\
p_{\text{error}} &= \sin^2 \alpha
\end{aligned}
$$

$$
\begin{aligned}
\text{(b)} \quad p_{\text{error}} &= p_1 + \sum_{i:\lambda_i<0} \lambda_i \\
&= \frac{1}{2} + \left\{ -\frac{1}{2} \cos(2\alpha) \right\} \\
&= \frac{1}{2} - \frac{1}{2} \cos(2\alpha) \\
p_{\text{error}} &= \sin^2 \alpha
\end{aligned}
$$

$$
\begin{aligned}
\text{(c)} \quad p_{\text{error}} &= \frac{1}{2} \{1 - \text{tr} \, |p_2\rho_2 - p_1\rho_1|\} \\
&= \frac{1}{2} \left\{ 1 - \left| -\frac{1}{2} \cos(2\alpha) \right| - \left| \frac{1}{2} \cos(2\alpha) \right| \right\} \\
&= \frac{1}{2} \{1 - \cos(2\alpha)\} \\
p_{\text{error}} &= \sin^2 \alpha
\end{aligned}
$$

e) Suppose Bob's POVM $\{E_i\}$ now has more than two possible outcomes. Let $p(1|i)$ and $p(2|i)$ be the probabilities that ρ_1 or ρ_2 was prepared, respectively, given that the measurement outcome is i.

Let $p_{\text{max}}{}^{(i)} \equiv \max\{p(1|i), p(2|i)\}$ and $p_{\text{min}}{}^{(i)} \equiv \min\{p(1|i), p(2|i)\}$. Note that $p_{\text{max}}{}^{(i)} + p_{\text{min}}{}^{(i)} = 1$ and $p_{\text{max}}{}^{(i)} - p_{\text{min}}{}^{(i)} = |p(2|i) - p(1|i)|$.

$$
\begin{aligned}
p_{\text{error}}(i) &= \min\{p(1|i), p(2|i)\} \\
&= p_{\text{min}}{}^{(i)} \\
&= \frac{1}{2} \left(p_{\text{min}}{}^{(i)} + p_{\text{min}}{}^{(i)} \right) \\
&= \frac{1}{2} \left(\left(1 - p_{\text{max}}{}^{(i)} \right) + \left(p_{\text{max}}{}^{(i)} - |p(2|i) - p(1|i)| \right) \right) \\
&= \frac{1}{2} (1 - |p(2|i) - p(1|i)|) \\
p_{\text{error}}(i) &= \frac{1}{2} - \frac{1}{2} |p(2|i) - p(1|i)|
\end{aligned}
$$

We can relate the following conditional probabilities:

$$\begin{aligned} p_i\, p(1|i) &= p_1\, p(i|1) &= p_1 \text{ tr}\, \rho_1 E_i \\ p_i\, p(2|i) &= p_2\, p(i|2) &= p_2 \text{ tr}\, \rho_2 E_i \end{aligned}$$

The total probability of error when using this POVM is given by averaging the conditional error probabilities over the measurement outcomes:

$$\begin{aligned} p_{\text{error}} &= \sum_i p_i\, p_{\text{error}}(i) \\ &= \sum_i p_i \left(\frac{1}{2} - \frac{1}{2} \left| p(2|i) - p(1|i) \right| \right) \\ &= \frac{1}{2} \sum_i p_i - \frac{1}{2} \sum_i \left| p_i\, p(2|i) - p_i\, p(1|i) \right| \\ &= \frac{1}{2} \cdot 1 - \frac{1}{2} \sum_i \left| p_2 \text{ tr } (\rho_2 E_i) - p_1 \text{ tr } (\rho_1 E_i) \right| \\ p_{\text{error}} &= \frac{1}{2} - \frac{1}{2} \sum_i \left| \text{tr } ((p_2 \rho_2 - p_1 \rho_1)\, E_i) \right| \end{aligned}$$

f) As before, we can diagonalize $p_2\rho_2 - p_1\rho_1 = \sum_a |a\rangle\lambda_a\langle a|$. The result from part e can then be rewritten as:

$$\begin{aligned} p_{\text{error}} &= \frac{1}{2} - \frac{1}{2} \sum_i \left| \text{tr} \left(\sum_a |a\rangle\lambda_a\langle a| E_i \right) \right| \\ p_{\text{error}} &= \frac{1}{2} - \frac{1}{2} \sum_i \left| \sum_a \lambda_a \langle a|E_i|a\rangle \right| \end{aligned}$$

The triangle inequality states that $\sum_a |c_a| \geq |\sum_a c_a|$. Alternatively, $-|\sum_a c_a| \geq -\sum_a |c_a|$.

$$\begin{aligned} p_{\text{error}} &\geq \frac{1}{2} - \frac{1}{2} \sum_i \sum_a \left| \lambda_a \langle a|E_i|a\rangle \right| \\ p_{\text{error}} &\geq \frac{1}{2} - \frac{1}{2} \sum_i \sum_a |\lambda_a|\, |\langle a|E_i|a\rangle| \end{aligned}$$

As argued in part b, $\langle a|E_i|a\rangle$ is bounded between zero and one $\forall\, a, i$.

$$p_{\text{error}} \geq \frac{1}{2} - \frac{1}{2} \sum_i \sum_a |\lambda_a|\, \langle a|E_i|a\rangle$$

$$
\begin{aligned}
&\geq \frac{1}{2} - \frac{1}{2}\sum_a |\lambda_a| \langle a| \sum_i E_i |a\rangle \\
&\geq \frac{1}{2} - \frac{1}{2}\sum_a |\lambda_a| \langle a|I|a\rangle \\
&\geq \frac{1}{2} - \frac{1}{2}\sum_a |\lambda_a| \\
&\geq \frac{1}{2} - \frac{1}{2}\ \mathrm{tr}\ |p_2\rho_2 - p_1\rho_1| \\
p_{\mathrm{error}} &\geq \frac{1}{2} - \frac{1}{2}\|p_2\rho_2 - p_1\rho_1\|_{\mathrm{tr}}
\end{aligned}
$$

In the next-to-last step, we note that the eigenvalues of $|p_2\rho_2 - p_1\rho_1|$ are $\{|\lambda_a|\}$. In the final step, we recognize that the trace norm of a Hermetian operator A is $\|A\|_{\mathrm{tr}} = \mathrm{tr}\left[\left(A^\dagger A\right)^{\frac{1}{2}}\right] = \mathrm{tr}\left[\left(A^2\right)^{\frac{1}{2}}\right] = \mathrm{tr}\ |A|$.

2.2 One-qubit decoherence

In this problem, we denote the identity I as σ_0. The Pauli matrices are denoted as $\sigma_1, \sigma_2, \sigma_3$. Note that σ_μ is Hermetian for $\mu = 0, 1, 2, 3$.

a) Let $\mathcal{E}$ be a quantum operation acting on the density operator ρ of a single qubit. This operation has an operator-sum representation similar to equation (3.22) in the section of the notes on generalized measurements.

$$
\mathcal{E}(\rho) = \sum_a M_a \rho M_a{}^\dagger
$$

The only difference for a quantum operation is that the operation elements $\{M_a\}$ do not need to form a partition of the identity (which implies that $\mathcal{E}$ does not need to preserve trace). We can expand each 2×2 matrix M_a into the basis $\{\sigma_\mu\}$:

$$
\begin{aligned}
\mathcal{E}(\rho) &= \sum_a \left(\sum_{\mu=0}^{3} c_{a\mu}\sigma_\mu\right) \rho \left(\sum_{\nu=0}^{3} c^*_{a\nu}\sigma_\nu{}^\dagger\right) \\
&= \sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\sum_a c_{a\mu}c^*_{a\nu}\sigma_\mu\rho\sigma_\nu \\
\mathcal{E}(\rho) &= \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \mathcal{E}_{\mu\nu}\sigma_\mu\rho\sigma_\nu
\end{aligned}
$$

where we define complex numbers $\mathcal{E}_{\mu\nu} \equiv \sum_a c_{a\mu} c^*_{a\nu}$.

Note that $\mathcal{E}^*_{\mu\nu} = \sum_a (c_{a\mu} c^*_{a\nu})^* = \sum_a c^*_{a\mu} c_{a\nu} = \sum_a c_{a\nu} c^*_{a\mu} = \mathcal{E}_{\nu\mu}$.

b) To require that the completely positive map $\mathcal{E}$ be trace-preserving (so that it becomes a superoperator mapping density operators to density operators), we add the constraint $\sum_a {M_a}^\dagger M_a = I = \sigma_0$.

$$\begin{aligned}
\sigma_0 &= \sum_a {M_a}^\dagger M_a \\
&= \sum_a \left(\sum_{\nu=0}^{3} c^*_{a\nu} {\sigma_\nu}^\dagger \right) \left(\sum_{\mu=0}^{3} c_{a\mu} \sigma_\mu \right) \\
&= \sum_{\nu=0}^{3} \sum_{\mu=0}^{3} \sum_a c_{a\mu} c^*_{a\nu} \sigma_\nu \sigma_\mu \\
&= \sum_{\nu=0}^{3} \sum_{\mu=0}^{3} \mathcal{E}_{\mu\nu} \sigma_\nu \sigma_\mu \\
&= \mathcal{E}_{00} \sigma_0 \sigma_0 + \sum_{j=1}^{3} \mathcal{E}_{j0} \sigma_0 \sigma_j + \sum_{k=1}^{3} \mathcal{E}_{0k} \sigma_k \sigma_0 + \sum_{k=1}^{3} \sum_{j=1}^{3} \mathcal{E}_{jk} \sigma_k \sigma_j \\
&= \mathcal{E}_{00} \sigma_0 + \sum_{j=1}^{3} \mathcal{E}_{j0} \sigma_j + \sum_{k=1}^{3} \mathcal{E}_{0k} \sigma_k + \sum_{k=1}^{3} \sum_{j=1}^{3} \mathcal{E}_{jk} \left(\delta_{kj} \sigma_0 + i \epsilon_{kjl} \sigma_l \right) \\
&= \mathcal{E}_{00} \sigma_0 + \sum_{j=1}^{3} \left(\mathcal{E}_{j0} + \mathcal{E}_{0j} \right) \sigma_j + \sum_{j=1}^{3} \mathcal{E}_{jj} \sigma_0 - i \sum_{k=1}^{3} \sum_{j=1}^{3} \mathcal{E}_{jk} \epsilon_{jkl} \sigma_l \\
\sigma_0 &= \sum_{\mu=0}^{3} \mathcal{E}_{\mu\mu} \sigma_0 + \sum_{j=1}^{3} \left(\mathcal{E}_{j0} + \mathcal{E}_{0j} \right) \sigma_j - i \sum_{j=1}^{3} \sum_{k=1}^{3} \mathcal{E}_{jk} \epsilon_{jkl} \sigma_l
\end{aligned}$$

where $\epsilon_{kjl} = -\epsilon_{jkl}$ is the anticommutative tensor of rank three.

Since $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ forms a basis for 2×2 matrices, we can read off four linearly independent conditions from the coefficients on each side:

$$\begin{aligned}
1 &= \mathcal{E}_{00} + \mathcal{E}_{11} + \mathcal{E}_{22} + \mathcal{E}_{33} \\
0 &= (\mathcal{E}_{10} + \mathcal{E}_{01}) - i \left(\mathcal{E}_{23} - \mathcal{E}_{32} \right) \\
0 &= (\mathcal{E}_{20} + \mathcal{E}_{02}) - i \left(\mathcal{E}_{31} - \mathcal{E}_{13} \right) \\
0 &= (\mathcal{E}_{30} + \mathcal{E}_{03}) - i \left(\mathcal{E}_{12} - \mathcal{E}_{21} \right)
\end{aligned}$$

Applying $\mathcal{E}_{\mu\nu} = \mathcal{E}^*_{\nu\mu}$, these conditions can be written as

$$\begin{aligned}
1 &= \mathcal{E}_{00} + \mathcal{E}_{11} + \mathcal{E}_{22} + \mathcal{E}_{33} \\
0 &= 2\,\Re(\mathcal{E}_{10}) + 2\,\Im(\mathcal{E}_{23}) \\
0 &= 2\,\Re(\mathcal{E}_{20}) + 2\,\Im(\mathcal{E}_{31}) \\
0 &= 2\,\Re(\mathcal{E}_{30}) + 2\,\Im(\mathcal{E}_{12})
\end{aligned}$$

where $\Re$ and $\Im$ denote the real and imaginary coefficients of a complex number (*e.g.*, if $z = a + bi$, then $\Re(z) = a$ and $\Im(z) = b$).

This is further simplified as:

$$\begin{aligned}
1 &= \mathcal{E}_{00} + \mathcal{E}_{11} + \mathcal{E}_{22} + \mathcal{E}_{33} \\
\Re(\mathcal{E}_{10}) &= \Im(\mathcal{E}_{32}) \\
\Re(\mathcal{E}_{20}) &= \Im(\mathcal{E}_{13}) \\
\Re(\mathcal{E}_{30}) &= \Im(\mathcal{E}_{21})
\end{aligned}$$

If these four conditions hold for complex numbers $\mathcal{E}_{\mu\nu}$ corresponding to a quantum operation $\mathcal{E}$ (as defined in part a), then the map is trace-preserving.

c) Suppose $\mathcal{E}$ is a linear map taking 2×2 Hermetian operators to Hermetian operators. We can determine how $\mathcal{E}$ acts on a general Hermetian operator by analyzing how it acts on each element of some basis for the space.

We can express a 2×2 Hermetian operator as

$$\rho(P) = \frac{1}{2} \sum_{\mu=0}^{3} P_\mu \sigma_\mu$$

where P_0, P_1, P_2, P_3 are real numbers.

The matrices $\{\sigma_\mu\}$ form a basis for the Hermetian operators (with real coefficients). The most general mapping of a basis element σ_μ under $\mathcal{E}$ is to some Hermetian operator $\rho(Q)$

$$\mathcal{E}(\sigma_\mu) = \rho(Q) = \frac{1}{2} \sum_{\nu=0}^{3} Q_\nu \sigma_\nu$$

where the Q_ν's are real numbers. Combining such statements for each basis element, we can write $\mathcal{E}(\sigma_\mu) = \sum_\nu M_{\mu\nu}\sigma_\nu$ where M is a 4×4 matrix with real matrix elements.

We can now apply linearity to derive the general relation:

$$\begin{aligned}
\mathcal{E}(\rho(P)) &= \mathcal{E}\left(\frac{1}{2}\sum_{\mu=0}^{3} P_\mu \sigma_\mu\right) \\
&= \frac{1}{2}\sum_{\mu=0}^{3} P_\mu \mathcal{E}(\sigma_\mu) \\
&= \frac{1}{2}\sum_{\mu=0}^{3} P_\mu \left(\sum_{\nu=0}^{3} M_{\mu\nu}\sigma_\nu\right) \\
&= \frac{1}{2}\sum_{\nu=0}^{3}\sum_{\mu=0}^{3} M_{\mu\nu} P_\mu \sigma_\nu \\
&= \frac{1}{2}\sum_{\nu=0}^{3} P'_\nu \sigma_\nu \\
\mathcal{E}(\rho(P)) &= \rho(P')
\end{aligned}$$

where we define four real numbers $P'_\nu = \sum_\mu M_{\mu\nu} P_\mu$.

We've demonstrated that $\mathcal{E}(\rho(P)) = \rho(P')$ with $P' = MP$. Since there are no restrictions on the components of the real vectors P and P', the real matrix M has $4^2 = 16$ free (real) parameters. The dimension of the space of linear maps taking 2×2 Hermetian operators to Hermetian operators is 16.

d) The trace of a 2×2 Hermetian operator $\rho(P)$ is given by

$$\begin{aligned}
\operatorname{tr}\rho(P) &= \operatorname{tr}\left(\frac{1}{2}\sum_{\mu=0}^{3} P_\mu \sigma_\mu\right) \\
&= \frac{1}{2}\sum_{\mu=0}^{3} P_\mu \operatorname{tr}\sigma_\mu \\
&= \frac{1}{2}\left(P_0 \operatorname{tr}\sigma_0 + P_1 \operatorname{tr}\sigma_1 + P_2 \operatorname{tr}\sigma_2 + P_3 \operatorname{tr}\sigma_3\right) \\
\operatorname{tr}\rho(P) &= P_0
\end{aligned}$$

because the Pauli matrices σ_1, σ_2, σ_3 are traceless.

Suppose now that $\mathrm{tr}\,\rho = 1$ and that the linear mapping $\mathcal{E}$ is trace-preserving. From above, we see that this requires $P_0 = P_0' = 1$. Applying the definition of P_ν' from part c, we find that

$$P_0' = \sum_{\mu=0}^{3} M_{0\mu} P_\mu = M_{00} P_0 + \sum_j M_{0j} P_j \ .$$

In order to satisfy $P_0 = P_0' = 1$ for arbitrary P_j, we require that $M_{00} = 1$ and $M_{01} = M_{02} = M_{03} = 0$. The mapping $\mathcal{E}$ then looks like

$$\mathcal{E}: \quad \begin{pmatrix} 1 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ P_1' \\ P_2' \\ P_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ M_{01} & M_{11} & M_{12} & M_{13} \\ M_{02} & M_{21} & M_{22} & M_{23} \\ M_{03} & M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} 1 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

If we define a real three-component $\vec{v} = [M_{01}, M_{02}, M_{03}]$ and redefine M as a real 3×3 matrix with components M_{ij} above, then the mapping can be encapsulated as $\vec{P'} = M\vec{P} + \vec{v}$. There are nine free parameters in M and three free parameters in $\vec{v}$, so the dimension of the space of such trace-preserving maps is $9 + 3 = 12$.

e) Given that $\mathrm{tr}\,\rho = 1$ we can express $\rho = \frac{1}{2}[\sigma_0 + \vec{P}\cdot\vec{\sigma}]$ where $\vec{P}$ is the Bloch polarization vector. We can plug this into the expression from part a:

$$\begin{aligned} \mathcal{E}(\rho) &= \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \mathcal{E}_{\mu\nu}\sigma_\mu\rho\sigma_\nu \\ &= \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \mathcal{E}_{\mu\nu}\sigma_\mu \left(\frac{1}{2}\left[\sigma_0 + \vec{P}\cdot\vec{\sigma}\right]\right)\sigma_\nu \\ \mathcal{E}(\rho) &= \frac{1}{2}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \mathcal{E}_{\mu\nu}\sigma_\mu\sigma_\nu + \frac{1}{2}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \mathcal{E}_{\mu\nu}\sigma_\mu\left(\vec{P}\cdot\vec{\sigma}\right)\sigma_\nu \end{aligned}$$

Under a trace-preserving map $\mathcal{E}$, we know from part d that

$$\mathcal{E}(\rho) = \rho' = \frac{1}{2}\left[\sigma_0 + \vec{P'}\cdot\vec{\sigma}\right] = \frac{1}{2}\left[\sigma_0 + \left(M\vec{P} + \vec{v}\right)\cdot\vec{\sigma}\right]$$

for some real M, $\vec{v}$. The components of $\vec{v}$ can be matched up with the piece of $\mathcal{E}(\rho)$ above that is independent of the polarization vector P.

$$\frac{1}{2}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\mathcal{E}_{\mu\nu}\sigma_\mu\sigma_\nu = \frac{1}{2}\left[\sigma_0 + \vec{v}\cdot\vec{\sigma}\right] = \frac{1}{2}\left[\sigma_0 + v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3\right]$$

This is closely related to the expression we worked with in part b (but with σ_μ and σ_ν swapped). By the same approach of expanding the left-hand side, we find the relations:

$$\begin{aligned} v_1 &= 2\,\Re(\mathcal{E}_{01}) + 2\,\Im(\mathcal{E}_{32}) \\ v_2 &= 2\,\Re(\mathcal{E}_{02}) + 2\,\Im(\mathcal{E}_{13}) \\ v_3 &= 2\,\Re(\mathcal{E}_{03}) + 2\,\Im(\mathcal{E}_{21}) \end{aligned}$$

But $\Im(\mathcal{E}_{32}) = \Im(\mathcal{E}_{23}^*) = -\Im(\mathcal{E}_{23}) = \Re(\mathcal{E}_{10}) = \Re(\mathcal{E}_{01})$, using the property $\mathcal{E}_{\mu\nu}^* = \mathcal{E}_{\nu\mu}$ and the results from part b. There are similar relations for the other quantities, so we find that

$$v_j = 4\,\Re(\mathcal{E}_{0j}) \text{ for } j \in \{1,2,3\}\,.$$

f) Now consider the space of trace-preserving linear maps taking $N \times N$ Hermitian operators to Hermitian operators. We can find the dimension of this space by following either the approach of parts b and c or the approach of part d for the 2×2 operators.

A basis for the $N \times N$ Hermitian operators has N^2 elements, so the most general linear transformation acting on the basis elements has $(N^2)^2 = N^4$ free real parameters. The restriction of being a trace-preserving map gives a number of independent conditions (like those found in part b) equal to the number of basis elements, namely N^2. Hence, the number of free real parameters for the trace-preserving maps is $N^4 - N^2$.

An alternative approach is to construct a real matrix M and real vector $\vec{v}$ as in part d, relating the "polarization" vectors $\vec{P}$ and $\vec{P}'$. The restriction to a trace-preserving map means that M is a $(N^2 - 1) \times (N^2 - 1)$ matrix and $\vec{v}$ has $(N^2 - 1)$ components. This gives $(N^2-1)^2 + (N^2-1) = N^4 - 2N^2 + 1 + N^2 - 1 = N^4 - N^2$ free real parameters.

2.3 Fidelity

a) Consider two operators R and S such that RS and SR are Hermetian. Then we can diagonalize each of RS and SR. Let $\{|\varphi_i\rangle\}$ be the set of eigenstates of RS, with corresponding eigenvalues $\{\lambda_i\}$.

$$\begin{aligned} RS|\varphi_i\rangle &= \lambda_i|\varphi_i\rangle \\ SRS|\varphi_i\rangle &= S\left(\lambda_i|\varphi_i\rangle\right) \\ SR\left(S|\varphi_i\rangle\right) &= \lambda_i\left(S|\varphi_i\rangle\right) \end{aligned}$$

If we define $|\psi_i\rangle \equiv S|\varphi_i\rangle$, then $SR|\psi_i\rangle = \lambda_i|\psi_i\rangle$. Hence, every eigenvalue of RS is also an eigenvalue of SR.

We can apply the same method to the eigenstates of SR to show that every eigenvalue of SR is also an eigenvalue of RS. Thus, RS and SR have the same set of eigenvalues (with the same multiplicities).

Now let $R = AB$ and let $S = BA$, where A and B are Hermetian operators.

Note first of all that $R^\dagger = (AB)^\dagger = B^\dagger A^\dagger = BA = S$ and similarly $S^\dagger = R$. Then $(RS)^\dagger = S^\dagger R^\dagger = RS$ and $(SR)^\dagger = R^\dagger S^\dagger = SR$. That is, RS and SR are Hermetian.

The argument above showed that $RS = ABBA$ and $SR = BAAB$ have the same set of eigenvalues $\{\lambda_i\}$. The trace norm evaluates to:

$$\|AB\|_{\text{tr}} = \text{tr}\sqrt{(AB)^\dagger (AB)} = \text{tr}\sqrt{BAAB} = \sum_i \sqrt{\lambda_i}$$

$$\|BA\|_{\text{tr}} = \text{tr}\sqrt{(BA)^\dagger (BA)} = \text{tr}\sqrt{ABBA} = \sum_i \sqrt{\lambda_i}$$

Then $\left\|\tilde{\rho}^{\frac{1}{2}}\rho^{\frac{1}{2}}\right\|_{\text{tr}} = \left\|\rho^{\frac{1}{2}}\tilde{\rho}^{\frac{1}{2}}\right\|_{\text{tr}}$, so the fidelity $F(\rho,\tilde{\rho}) = F(\tilde{\rho},\rho)$ is symmetric in its two arguments.

b) The overlap of the probability distributions from trying to distinguish states ρ_1 and ρ_2 by a POVM $\{E_i\}$ is given by:

$$\text{Overlap}(\rho,\tilde{\rho};\{E_i\}) \equiv \sum_i \sqrt{\text{tr}\,\rho E_i}\cdot\sqrt{\text{tr}\,\tilde{\rho}E_i}$$

The E_i's are a set of Hermetian operators that sum to the identity. Furthermore, they have real nonnegative eigenvalues, so we can define $(E_i)^{\frac{1}{2}}$ to also be a nonnegative Hermetian operator. Similarly, density operators ρ and $\tilde{\rho}$ can be expressed as the squares of nonnegative Hermetian operators.

Let $A_i = \rho^{\frac{1}{2}} E_i{}^{\frac{1}{2}}$ and $B_i = U\tilde{\rho}^{\frac{1}{2}} E_i{}^{\frac{1}{2}}$, where U is an arbitrary unitary operator. We can calculate various inner products:

$$\begin{aligned}
(A_i, A_i) &= \text{tr}\left(A_i{}^\dagger A_i\right) \\
&= \text{tr}\left(\left(\rho^{\frac{1}{2}} E_i{}^{\frac{1}{2}}\right)^\dagger \rho^{\frac{1}{2}} E_i{}^{\frac{1}{2}}\right) \\
&= \text{tr}\left(E_i{}^{\frac{1}{2}} \rho^{\frac{1}{2}} \rho^{\frac{1}{2}} E_i{}^{\frac{1}{2}}\right) \\
(A_i, A_i) &= \text{tr}\left(\rho E_i\right)
\end{aligned}$$

$$\begin{aligned}
(B_i, B_i) &= \text{tr}\left(A_i{}^\dagger A_i\right) \\
&= \text{tr}\left(\left(U\tilde{\rho}^{\frac{1}{2}} E_i{}^{\frac{1}{2}}\right)^\dagger U\tilde{\rho}^{\frac{1}{2}} E_i{}^{\frac{1}{2}}\right) \\
&= \text{tr}\left(E_i{}^{\frac{1}{2}} \tilde{\rho}^{\frac{1}{2}} U^\dagger U \tilde{\rho}^{\frac{1}{2}} E_i{}^{\frac{1}{2}}\right) \\
&= \text{tr}\left(E_i{}^{\frac{1}{2}} \tilde{\rho}^{\frac{1}{2}} \tilde{\rho}^{\frac{1}{2}} E_i{}^{\frac{1}{2}}\right) \\
(B_i, B_i) &= \text{tr}\left(\tilde{\rho} E_i\right)
\end{aligned}$$

$$\begin{aligned}
(A_i, B_i) &= \text{tr}\left(A_i{}^\dagger B_i\right) \\
&= \text{tr}\left(\left(\rho^{\frac{1}{2}} E_i{}^{\frac{1}{2}}\right)^\dagger U\tilde{\rho}^{\frac{1}{2}} E_i{}^{\frac{1}{2}}\right) \\
&= \text{tr}\left(E_i{}^{\frac{1}{2}} \rho^{\frac{1}{2}} U \tilde{\rho}^{\frac{1}{2}} E_i{}^{\frac{1}{2}}\right) \\
(A_i, B_i) &= \text{tr}\left(\rho^{\frac{1}{2}} U \tilde{\rho}^{\frac{1}{2}} E_i\right)
\end{aligned}$$

We can apply the triangle inequality to the last relation:

$$\begin{aligned}
\sum_i |(A_i, B_i)| &\geq \left|\sum_i (A_i, B_i)\right| \\
&\geq \left|\sum_i \text{tr}\left(\rho^{\frac{1}{2}} U \tilde{\rho}^{\frac{1}{2}} E_i\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&\geq \left| \text{tr} \left(\rho^{\frac{1}{2}} U \tilde{\rho}^{\frac{1}{2}} \sum_i E_i \right) \right| \\
\sum_i |(A_i, B_i)| &\geq \left| \text{tr} \left(\rho^{\frac{1}{2}} U \tilde{\rho}^{\frac{1}{2}} \right) \right|
\end{aligned}
$$

Next we apply the Schwarz inequality:

$$
\begin{aligned}
\sqrt{(A_i, A_i)} \sqrt{(B_i, B_i)} &\geq |(A_i, B_i)| \\
\sum_i \sqrt{(A_i, A_i)} \sqrt{(B_i, B_i)} &\geq \sum_i |(A_i, B_i)| \\
\sum_i \sqrt{\text{tr } (\rho E_i)} \sqrt{\text{tr } (\tilde{\rho} E_i)} &\geq \left| \text{tr} \left(\rho^{\frac{1}{2}} U \tilde{\rho}^{\frac{1}{2}} \right) \right|
\end{aligned}
$$

Thus, $\text{Overlap}(\rho, \tilde{\rho}; \{E_i\}) \geq \left| \text{tr} \left(\rho^{\frac{1}{2}} U \tilde{\rho}^{\frac{1}{2}} \right) \right|$.

c) We can apply the polar decomposition theorem to write

$$
\begin{aligned}
\tilde{\rho}^{\frac{1}{2}} \rho^{\frac{1}{2}} &= V \sqrt{\left(\tilde{\rho}^{\frac{1}{2}} \rho^{\frac{1}{2}} \right)^{\dagger} \left(\tilde{\rho}^{\frac{1}{2}} \rho^{\frac{1}{2}} \right)} \\
&= V \sqrt{\rho^{\frac{1}{2}} \tilde{\rho}^{\frac{1}{2}} \tilde{\rho}^{\frac{1}{2}} \rho^{\frac{1}{2}}} \\
\tilde{\rho}^{\frac{1}{2}} \rho^{\frac{1}{2}} &= V \sqrt{\rho^{\frac{1}{2}} \tilde{\rho} \rho^{\frac{1}{2}}}
\end{aligned}
$$

where V is some particular unitary operator.

The relation from part b becomes:

$$
\begin{aligned}
\text{Overlap}(\rho, \tilde{\rho}; \{E_i\}) &\geq \left| \text{tr} \left(\rho^{\frac{1}{2}} U \tilde{\rho}^{\frac{1}{2}} \right) \right| \\
&\geq \left| \text{tr} \left(U \tilde{\rho}^{\frac{1}{2}} \rho^{\frac{1}{2}} \right) \right| \\
\text{Overlap}(\rho, \tilde{\rho}; \{E_i\}) &\geq \left| \text{tr} \left(U V \sqrt{\rho^{\frac{1}{2}} \tilde{\rho} \rho^{\frac{1}{2}}} \right) \right|
\end{aligned}
$$

The unitary operator U was arbitrary when introduced in part b. We can choose to set $U = V^{-1}$, and $\text{Overlap}(\rho, \tilde{\rho}; \{E_i\}) \geq \left| \text{tr} \sqrt{\rho^{\frac{1}{2}} \tilde{\rho} \rho^{\frac{1}{2}}} \right|$. By defining the square root of the operator under the radical to have nonnegative eigenvalues, we can drop the absolute values:

$$
\text{Overlap}(\rho, \tilde{\rho}; \{E_i\}) \geq \text{tr} \sqrt{\rho^{\frac{1}{2}} \tilde{\rho} \rho^{\frac{1}{2}}}
$$

d) We can actually calculate the fidelity of two single qubit states with polarization vectors $\vec{P}$ and $\vec{Q}$ without explicitly finding the eigenvalues of any operator (calculating trace and determinant will be sufficient).

As an aside, however, we can find the general relation of eigenvalues to the trace and determinant of a matrix. In particular, let $\{\lambda_1, \lambda_2\}$ be the eigenvalues of a 2×2 matrix M. Then $\det M = \lambda_1 \lambda_2$ and $\operatorname{tr} M = \lambda_1 + \lambda_2$. Solving for $\lambda_1 = \operatorname{tr} M - \lambda_2$ and plugging into the determinant equation gives the quadratic relation:

$$\lambda_2^2 - \lambda_2 (\operatorname{tr} M) + \det M = 0$$

Hence, the eigenvalues of M are $\left\{\frac{1}{2}\left(\operatorname{tr} M \pm \sqrt{(\operatorname{tr} M)^2 - 4 \det M}\right)\right\}$.

We are interested in calculating the fidelity $F\left(\rho(\vec{P}), \rho(\vec{Q})\right)$. Let $A = \rho(\vec{P})^{\frac{1}{2}} \rho(\vec{Q}) \rho(\vec{P})^{\frac{1}{2}}$. Note that $A^\dagger = A$, so we can choose a diagonal basis for this Hermetian operator and express $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Furthermore, from part c, we know that $\rho(\vec{P})^{\frac{1}{2}} \rho(\vec{Q}) \rho(\vec{P})^{\frac{1}{2}}$ is nonnegative, so we can define $\sqrt{A} = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}$.

Next, we calculate the fidelity:

$$\begin{aligned}
F\left(\rho(\vec{P}), \rho(\vec{Q})\right) &= \left(\operatorname{tr} \sqrt{\rho(\vec{P})^{\frac{1}{2}} \rho(\vec{Q}) \rho(\vec{P})^{\frac{1}{2}}}\right)^2 \\
&= \left(\operatorname{tr} \sqrt{A}\right)^2 \\
&= \left(\sqrt{\lambda_1} + \sqrt{\lambda_2}\right)^2 \\
&= \lambda_1 + \lambda_2 + 2\sqrt{\lambda_1 \lambda_2} \\
F\left(\rho\left(\vec{P}\right), \rho\left(\vec{Q}\right)\right) &= \operatorname{tr} A + 2\sqrt{\det A}
\end{aligned}$$

We need only evalate the trace and determinant of A to find the fidelity. Using the Bloch parametrization $\rho(\vec{P}) = \frac{1}{2}[I + \vec{\sigma} \cdot \vec{P}]$, we find that

$$\operatorname{tr} A = \operatorname{tr} \left(\rho(\vec{P})^{\frac{1}{2}} \rho(\vec{Q}) \rho(\vec{P})^{\frac{1}{2}}\right)$$

$$
\begin{aligned}
&= \text{tr}\left(\rho(\vec{P})^{\frac{1}{2}}\rho(\vec{P})^{\frac{1}{2}}\rho(\vec{Q})\right) \\
&= \text{tr}\left(\rho(\vec{P})\rho(\vec{Q})\right) \\
&= \text{tr}\left(\frac{1}{2}\left[I+\vec{\sigma}\cdot\vec{P}\right]\frac{1}{2}\left[I+\vec{\sigma}\cdot\vec{Q}\right]\right) \\
\text{tr}\,A &= \frac{1}{4}\,\text{tr}\left(I+\vec{\sigma}\cdot\vec{P}+\vec{\sigma}\cdot\vec{Q}+\left(\vec{\sigma}\cdot\vec{P}\right)\left(\vec{\sigma}\cdot\vec{Q}\right)\right)
\end{aligned}
$$

Note that the Pauli operators $\sigma_1, \sigma_2, \sigma_3$ are traceless, so the middle two terms vanish under trace. Also, the last term contributes only when the σ matrices multiply to the identity (namely ${\sigma_1}^2$, ${\sigma_2}^2$, and ${\sigma_3}^2$). Thus, $\text{tr}\,A = \frac{1}{4}\,\text{tr}\,(I + P_1Q_1I + P_2Q_2I + P_3Q_3I) = \frac{1}{2}\left(1+\vec{P}\cdot\vec{Q}\right)$.

Next we calculate the determinant:

$$
\begin{aligned}
\det A &= \det\left(\rho(\vec{P})^{\frac{1}{2}}\rho(\vec{Q})\rho(\vec{P})^{\frac{1}{2}}\right) \\
&= \left(\det\rho(\vec{P})^{\frac{1}{2}}\right)\left(\det\rho(\vec{Q})\right)\left(\det\rho(\vec{P})^{\frac{1}{2}}\right) \\
&= \left(\det\rho(\vec{P})\right)\left(\det\rho(\vec{Q})\right) \\
\det A &= \det\left(\frac{1}{2}\left[I+\vec{\sigma}\cdot\vec{P}\right]\right)\cdot\det\left(\frac{1}{2}\left[I+\vec{\sigma}\cdot\vec{Q}\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
\det\left(\frac{1}{2}\left[I+\vec{\sigma}\cdot\vec{P}\right]\right) &= \det\left[\frac{1}{2}\begin{pmatrix}1+P_3 & P_1-iP_2\\ P_1+iP_2 & 1-P_3\end{pmatrix}\right] \\
&= \left(\frac{1}{2}\right)^2\det\begin{pmatrix}1+P_3 & P_1-iP_2\\ P_1+iP_2 & 1-P_3\end{pmatrix} \\
&= \frac{1}{4}\left((1-{P_3}^2)-({P_1}^2+{P_2}^2)\right) \\
\det\left(\frac{1}{2}\left[I+\vec{\sigma}\cdot\vec{P}\right]\right) &= \frac{1}{4}\left(1-\vec{P}^2\right)
\end{aligned}
$$

Then $\det A = \frac{1}{16}\left(1-\vec{P}^2\right)\left(1-\vec{Q}^2\right)$.

Hence, the fidelity is given by:

$$
\begin{aligned}
F\left(\rho(\vec{P}),\rho(\vec{Q})\right) &= \text{tr}\,A + 2\sqrt{\det A} \\
&= \frac{1}{2}\left(1+\vec{P}\cdot\vec{Q}\right)+2\sqrt{\frac{1}{16}\left(1-\vec{P}^2\right)\left(1-\vec{Q}^2\right)} \\
F\left(\rho(\vec{P}),\rho(\vec{Q})\right) &= \frac{1}{2}\left(1+\vec{P}\cdot\vec{Q}+\sqrt{(1-\vec{P}^2)(1-\vec{Q}^2)}\right)
\end{aligned}
$$