1. Light bulb in a refrigerator

A refrigerator that draws 50 W of power is contained in a room at temperature 300ºK. A 100 W light bulb is left burning inside the refrigerator. Find the steady-state temperature inside the refrigerator assuming it operates reversibly and is perfectly insulated.

2. One refrigerator inside another

A (not necessarily reversible) air conditioner cools a room to temperature $\tau_m < \tau_h$, where $\tau_h$ is the temperature outdoors. Meanwhile, a (not necessarily reversible) refrigerator in the room cools drinks to temperature $\tau_l < \tau_m$. Thus the refrigerator and air conditioner, working together in series, move heat from the inside of the refrigerator to outdoors — the refrigerator does work $W_1$ to remove heat $Q_l$ from inside the refrigerator, while releasing heat $Q_m$ into the room, and the air conditioner does work $W_2$ to remove the heat $Q_m$ from the room, while releasing $Q_h$ outdoors. The refrigerator has coefficient of performance $\gamma_1 = Q_l/W_1$, and the air conditioner has coefficient of performance $\gamma_2 = Q_m/W_2$.

(a) Draw a diagram indicating the work done and the heat flow produced by the refrigerator and air conditioner. Your diagram should display three reservoirs at temperatures $\tau_l$, $\tau_m$, and $\tau_h$ — show the work done by the refrigerator and air conditioner, and the quantities of heat $Q_l$, $Q_m$, and $Q_h$. Use arrows to indicate clearly the direction of the work done and the heat flow.

(b) The combined system of refrigerator and air conditioner uses work $W_1 + W_2$ to move heat $Q_l$ from inside the refrigerator to outdoors. Express the reciprocal of the coefficient of performance $\gamma_{comb}^{-1}$ for this combined system in terms of $\gamma_1^{-1}$ and $\gamma_2^{-1}$. (You will need to do some algebra to find an expression for $\gamma_{comb}^{-1}$ in terms of $\gamma_1^{-1}$ and $\gamma_2^{-1}$, involving no other variables. This algebra is a bit easier if you work with the reciprocals of the coefficients of performance instead of the coefficients of performance themselves.)
(c) Suppose now that both the refrigerator and the air conditioner operate reversibly; hence each achieves the ideal Carnot coefficient of performance. Use your answer from part (b) to express $\gamma_{\text{comb}}^{-1}$ in terms of the temperatures of the reservoirs $\tau_l$, $\tau_m$, and $\tau_h$. Explain why your answer makes sense.

3. Photonic heat engine

Consider a heat engine undergoing a Carnot cycle, where the working fluid is a photon gas rather than a classical ideal gas. In the first stroke the gas expands isothermally at temperature $\tau_h$ from the initial volume $V_1$ to the final volume $V_2$. In the second stroke it expands isentropically to volume $V_3$, cooling to temperature $\tau_l$. In the third stroke it is compressed isothermally at temperature $\tau_l$ to volume $V_4$, and in the fourth stroke it is compressed isentropically back to volume $V_1$, heating to temperature $\tau_h$.

(a) The energy per unit volume of a photon gas is $U/V = A\tau^4$, where $A = \pi^2/15h^3c^3$. Use the thermodynamic identity

$$dU = \tau d\sigma - PdV$$

to find the entropy $\sigma$ of the gas, expressed in terms of $A$, $\tau$, and $V$. Assume that the entropy is zero at $\tau = 0$.

(b) Use the thermodynamic identity again to express the pressure $P$ in terms of $A$, $\tau$, and $V$.

(c) Calculate the work done $W_{12}$ and the heat added $Q_{12}$ during the first stroke of the cycle, expressed in terms of $A$, $\tau_h$, $V_1$ and $V_2$. Verify that $Q_{12} - W_{12}$ is the change in the internal energy of the gas.

(d) Express the work $W_{34}$ done by the gas in the third stroke (a negative number), in terms of $A$, $\tau_l$, $V_3$ and $V_4$.

(e) Use the condition $\sigma = \text{constant}$ during the isentropic strokes to express $V_3$ and $V_4$ in terms of $\tau_h$, $\tau_l$, $V_1$, and $V_2$.

(f) Find the work $W_{23}$ done during the second stroke and the work $W_{41}$ done during the fourth stroke.

(g) Express the net work $W = W_{12} + W_{23} + W_{34} + W_{41}$ done during the complete cycle in terms of $A$, $\tau_h$, $\tau_l$, $V_1$ and $V_2$. Comparing to $Q_{12}$, check that the engine achieves the ideal Carnot efficiency.
4. Bose condensation in two dimensions

Consider an ideal gas of non-relativistic spin-0 bosons, at temperature $\tau$, in a two-dimensional box of side $L$.

(a) Find the two-dimensional density of states factor $D(\varepsilon)$.

(b) Express the activity $\lambda \equiv e^{\mu/T}$ in terms of $N_0$, the number of particles in the ground orbital. Use the convention that the energy of the ground orbital is $\epsilon_0 = 0$.

(c) Find $N_e(\tau)$, the number of particles in excited orbitals. You may assume that the box is big enough so that the sum over states can be replaced by an integral. Be sure to use the formula found in (b) for $\lambda$, not the $N_0 \to \infty$ limit of that formula. Your answer for $N_e$ will therefore be expressed in terms of $N_0$. **Hint:** $\int dx (ae^x - 1) = \ln(a - e^{-x})$.

(d) Find the two-dimensional Einstein condensation temperature $\tau_E$. This is the smallest temperature such that, for $\tau > \tau_E$, the fraction $N_0/(N_0 + N_e)$ of particles in the ground orbital vanishes in the limit $L \to \infty$. (The limit is to be taken with the density $(N_0 + N_e)/L^2$ held fixed.)

5. Heat capacity of graphene

Geim and Novoselov received the 2010 Nobel Prize in Physics for their studies of graphene, a single layer of carbon atoms bonded into a two-dimensional hexagonal lattice. Remarkably, electrons in graphene behave like relativistic massless fermions; for each value of the wavenumber $\vec{k} = (k_x, k_y)$, there are two single-particle orbitals, with energies

$$\epsilon_{\pm}(\vec{k}) = \pm \hbar v |\vec{k}|.$$  

The Fermi energy is $\epsilon_F = 0$; hence at zero temperature the orbitals with negative energy are occupied, and the orbitals with positive energy are empty.

Assuming the electrons can be treated at an ideal gas, and that there are two spin states for each orbital, the internal energy of the electrons has the form

$$U(\tau) - U(0) = \frac{1}{3} \gamma A \tau^3,$$

where $A$ denotes the area, and hence the electron heat capacity is $C = A \tau^2$. Find $\gamma$. (**Hint:** $\int_0^{\infty} dx \, x^2/(e^x + 1) = 1.803$.)