Notes on noise

Gaussian phase noise

If decoherence is driven by a weak coupling of the system S to each of many "fluctuators" in the bath B, then it is reasonable to suppose that the fluctuations of B obey Gaussian statistics. Furthermore, in many systems "dephasing" is much stronger than "relaxation." For example, consider a qubit such that the basis states $|0\rangle$ and $|1\rangle$, are eigenstates of the unperturbed system Hamiltonian H_S, with energy splitting $\hbar\omega_{01}$. Low-frequency fluctuations ($\omega \ll \omega_{01}$) can cause a superposition to decohere in the $\{|0\rangle, |1\rangle\}$ basis, without driving transitions between $|0\rangle$ and $|1\rangle$.

To model a Gaussian dephasing process, consider the Hamiltonian

$$H = -\frac{1}{2}\omega_{01}\sigma_z - \frac{1}{2}f(t)\sigma_z \ , \tag{1}$$

where $f(t)$ is a fluctuating "magnetic field" that will be described stochastically (we assume $\hbar = 1$). For now, then, the "bath" is being treated as a fluctuating classical variable — later we will allow the bath to be a fluctuating quantum reservoir coupled to the system S. In this model, the system Hamiltonian $H_S = \frac{1}{2}\omega_{01}\sigma_z$ commutes with the system-bath coupling $H_{SB} = \frac{1}{2}f(t)\sigma_z$, and in fact we can transform H_S away by going to the interaction picture.

We denote averaging over the ensemble of functions $\{f\}$ as $[\cdot]_f$, and f is assumed to be a stationary (i.e., time-translation invariant) Gaussian random variable with mean zero, $[f(t)]_f = 0$, and covariance $K(t-t') \equiv [f(t)f(t')]_f$. Correlation functions for $f(t)$ are generated by

$$Z[J] \equiv \left[e^{\int dt J(t) f(t)}\right]_f = \exp\left(\frac{1}{2}\int dt dt' J(t)K(t-t')J(t')\right) \ . \tag{2}$$

An initial density operator $\rho(0)$ evolves in time T to

$$\rho(T) = \left[\exp\left(i\int_o^T \frac{1}{2}f(t)\sigma_z\right)\rho(0)\exp\left(-i\int_o^T \frac{1}{2}f(t)\sigma_z\right)\right]_f \ . \tag{3}$$

This has no effect on $|0\rangle\langle 0|$ or $|1\rangle\langle 1|$, but causes the coefficient of the off-diagonal entries $|0\rangle\langle 1|$ and $|1\rangle\langle 0|$ to decay by the factor

$$\exp\left(-\frac{1}{2}\int_0^T dt \int_0^T dt' K(t-t')\right) = \exp\left(-\frac{1}{2}\int_0^T dt\int_0^T dt' \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\tilde{K}(\omega)\right) \ , \tag{4}$$

where we have introduced the Fourier transform $\tilde{K}(\omega)$ of the covariance $K(t)$, which is said to be the "spectral density" or "power spectrum" of the noise. Doing the t and t' integrals we obtain

$$\exp\left(-\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tilde{K}(\omega)W_T(\omega)\right) \tag{5}$$

where the smooth window function $W_T(\omega)$ is

$$W_T(\omega) = \left|\int_0^T dt \ e^{-i\omega t}\right|^2 = \frac{4}{\omega^2}\sin^2(\omega T/2) \ , \tag{6}$$

which has most of its support on the interval $[0, 2\pi/T]$.

Assuming that $\tilde{K}(\omega = 0)$ is finite, we expect that for T sufficiently large, $\tilde{K}(\omega)$ can be regarded as approximately constant in the region where $W_T(\omega)$ is supported. Using $\int_{-\infty}^{\infty} dx \frac{\sin^2 x}{x^2} = \pi$, we then obtain $e^{-\Gamma_2 T}$, where the dephasing rate Γ_2 is

$$\Gamma_2 = \tilde{K}(\omega = 0) \ . \tag{7}$$

(Here we've assumed that $\tilde{K}(\omega)$ is continuous at $\omega = 0$ — otherwise we should average its limiting values as ω approaches zero from positive and negative values.) If the spectral density is flat ("white noise"), this formula for Γ_2 applies at any time T, but in general, the time scale for which dephasing can be described by a rate Γ_2 depends on the shape of the noise's spectral density.

Crudely speaking, we expect $\tilde{K}(\omega)$ to be roughly constant in the interval $[0, \omega_c]$, where $\omega_c = 2\pi/\tau_c$, and τ_c is a characteristic "autocorrelation" or "memory" time of the noise. That is, τ_c is chosen so that the correlation function $K(t - t')$ is small for $|t - t'| \gg \tau_c$. Thus we see that in order to speak of a "dephasing rate" Γ_2 (and a corresponding dephasing time $T_2 = \Gamma_2^{-1}$) we must consider evolution that has been "coarse-grained" in time. For the purpose of describing evolution over a time period $T \gg \tau_c$, the non-Markovian noise model can be replaced by a corresponding effective Markovian model in which the memory of the fluctuations can be neglected. But for $T \ll \tau_c$ such a description is not applicable.

Qubits as noise spectrometers

We see that an experimentalist, by measuring the dephasing time T_2 of a qubit, can probe the low-frequency noise power. In fact, noise power as a function of frequency is measurable, if the experimenter can vary the energy difference ω_{01} between the two computational basis states of the qubit, and observe how the polarization and relaxation time T_1 of the qubit depend on ω_{01}.

We will give a more careful account below of interaction-picture perturbation theory for a qubit coupled to a quantized bath, but for now, in the spirit of the above discussion of dephasing, consider relaxation driven by a fluctuating magnetic field in the $\hat{x}$ direction, as described by the Hamiltonian

$$H = -\frac{1}{2}\omega_{01}\sigma_z + f(t)\sigma_x \ . \tag{8}$$

If the fluctuating field $f(t)$ is weak, we may compute the probability for the qubit to make a transition from the excited state $|1\rangle$ to the ground state $|0\rangle$ during the time interval $[0, T]$, using the lowest nontrivial order of perturbation theory; after averaging over the fluctuating field we find

$$\begin{aligned} \mathrm{Prob}(1 \to 0) &= \left[\left| -i \int_0^T dt \ f(t) e^{-i\omega_{01}} \langle 0|\sigma_x|1\rangle \right|^2 \right]_f \\ &= \int_0^T dt \int_0^T dt' \ e^{-i\omega_{01}(t-t')} \left[f(t) f(t') \right]_f \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{K}(\omega) W_T(\omega - \omega_{01}) \ . \end{aligned} \tag{9}$$

This is similar to our expression for the dephasing probability, except that now the center of the window function has been shifted to the frequency ω_{01} of the transition.

Once again, if we consider the observation time T to be large compared to the autocorrelation time τ_c of the bath, then the support of the window function is narrow, and $\tilde{K}(\omega)$ is approximately constant in the window. Thus, after a suitable coarse-graining of the time evolution, we may identify a rate for the decay of the qubit

$$\Gamma_\downarrow = \tilde{K}(\omega = \omega_{01}) \ . \tag{10}$$

Similarly, for the transition from ground state to excited state, we find

$$\Gamma_\uparrow = \tilde{K}(\omega = -\omega_{01}) \ . \tag{11}$$

Thus negative frequency noise transfers energy from the noise reservoir to the qubit, exciting the qubit, while positive frequency noise transfers energy from qubit to the noise reservoir, returning the excited qubit to the ground state. Dephasing of a qubit, on the other hand, involves a negligible exchange of energy and therefore is controlled by low frequency noise.

Actually, for the case we have considered in which the noise source is classical, $f(t)$ and $f(t')$ are real commuting variables; therefore $K(t)$ is an even function of t and correspondingly $\tilde{K}(\omega)$ is an even function of ω. Classical noise is spectrally symmetric, and the rates for excitation and decay are equal.

On the other hand, noise driven by a quantized thermal bath is spectrally asymmetric. When the qubit comes to thermal equilibrium with the bath, up and down transitions occur at equal rates. If p_0 denotes the probability that the qubit is in the ground state $|0\rangle$ and p_1 denotes the probability that the qubit is in the excited state $|1\rangle$, then in equilibrium

$$p_0\Gamma_\uparrow = p_1\Gamma_\downarrow \Rightarrow \frac{\tilde{K}(-\omega_{01})}{\tilde{K}(\omega_{01})} = \frac{p_1}{p_0} = e^{-\beta\omega} \ ; \tag{12}$$

the ratio of noise strengths at positive and negative frequencies is given (for a thermal bath) by a Boltzmann factor, which is known as the Kubo-Martin-Schwinger (KMS) condition. The noise becomes classical in the high-temperature limit $\beta\omega \to 0$, and is in the deeply quantum regime for $\beta\omega \gg 1$. In optical transitions such quantum noise has been routinely studied for decades, but it is only relatively recently that electrical circuits have entered the quantum regime (e.g., for a 1 GHz resonator, the temperature is required to be below 20 mK).

For an "artificial atom" such as an electron spin in a quantum dot or the charge in a Cooper-pair box, the energy splitting ω_{01} can be tuned by the experimentalist. Then the equilibrium polarization of the qubit, observed as a function of ω_{01}, determines the ratio $\tilde{K}(-\omega)/\tilde{K}(\omega)$ (and hence the effective noise temperature of the bath). Furthermore, the sum $\tilde{K}(\omega)+\tilde{K}(-\omega)$ can be determined by observing how rapidly the polarization relaxes to its equilibrium value. The probability that the state $|0\rangle$ is occupied evolves according to

$$dp_0/dt = \Gamma_\downarrow p_1 - \Gamma_\uparrow p_0 \ ; \tag{13}$$

If Δp_0 is the deviation of p_0 from its equilibrium value (and therefore $-\Delta p_0$ is the deviation of p_1 from its equilibrium value), then

$$d\Delta p_0/dt = -\Gamma_1 \Delta p_0 \ , \tag{14}$$

where

$$\Gamma_1 = \Gamma_\downarrow + \Gamma_\uparrow = \tilde{K}(\omega) + \tilde{K}(-\omega) \ . \tag{15}$$

The time $T_1 = \Gamma_1^{-1}$ is the "relaxation time" of the qubit. Measuring the equilibrium polarization and the relaxation time determines both $\tilde{K}(\omega)$ and $\tilde{K}(-\omega)$.

White noise, random walk noise, and $1/f$ noise

If the spectral density is flat, then the covariance of the noise is a delta function in time:

$$[f(t)f(t')]_f \equiv K(t-t') = \kappa\delta(t-t') \ . \tag{16}$$

Thus we can generate a sample of white noise by drawing a value of $f(t)$ from a mean-zero Gaussian ensemble independently in each time bin (there are no correlations between $f(t)$ and $f(t')$ for $t \neq t'$). Though $f(t)$ has a well-defined mean (namely zero) if we average over a long time period, its instantaneous value is ill-defined (it is discontinuous at every point). Strictly speaking white noise is unphysical because there is an infinite amount of spectral power at high frequency. Nevertheless, white noise often provides a good description of physical noise sources; we just need to remember that some kind of high-frequency cut off is always required. For example, thermal Johnson noise in electrical circuits is white noise, at frequencies such that $\hbar\omega \ll kT$ so that each thermally populated mode in a wire carries energy kT. And as we have already discussed, noise can be well approximated by white noise if we coarse-grain in time by observing dynamics for a time large compared to the autocorrelation time of the noise.

A random walk (RW) is a noise process whose derivative is white noise (WN) — each step in the walk is independent of previous and subsequent steps. Since differentiating $f(t)$ is equivalent to dividing its Fourier transform $\tilde{f}(\omega)$ by ω, we have $f_{\rm RW}(\omega) = \omega^{-1} f_{\rm WN}(\omega)$ and $\tilde{K}_{\rm RW}(\omega) = \omega^{-2}\tilde{K}_{\rm WN}(\omega) \propto 1/\omega^2$; random walk noise has spectral density proportional to $1/\omega^2$. In this case, the integrated power spectrum diverges at low frequency, and correspondingly a random walk has no well-defined mean even if we average over a long time period — the walk wanders arbitrarily far away from the origin.

Intermediate between white noise and random walk noise is "$1/f$ noise" or "flicker noise" — what we might describe as the "half integral" of white noise in the sense that $\tilde{f} = \omega^{-1/2}\tilde{f}(\omega)$, which means that the power spectrum is $\tilde{K}(\omega) \propto 1/\omega$. The integrated spectral power of $1/f$ noise diverges at both low and high frequency, but only logarithmically, much milder than the ultraviolet divergence of white noise or the infrared divergence of random walk noise. The characteristic feature of $1/f$ noise is *scale invariance*. It has the same features on all time scales; i.e., the integrated spectral power is the same in each decade of frequency.

Such scale invariant $1/f$ noise seems to arise naturally in many settings (it applies accurately to the stock market, earthquakes, and classical music for example), and it is not so clear why this should be so. Scale invariance is a characteristic property of various phase transitions driven by thermal or quantum fluctuations, but in these cases it is usually necessary to tune one or more parameters to reach the scaling regime. In contrast, $1/f$ noise can arise generically, without tuning of parameters (which is sometimes referred to as "self-organized criticality").

In particular, $1/f$ noise often occurs in electrical circuits, including the superconducting circuits that are of interest for potentially scalable quantum computing, and can be the dominant source of noise at low frequency. If the phase noise spectrum behaves like $\tilde{K}(\omega) \propto 1/\omega$ at low frequency, then the dephasing cannot be characterized by a rate Γ even at large times (as one might expect given that the noise has no well defined mean). Rather, phase noise with a frequency comparable to the inverse running time of an experiment can generate a variability in the energy splitting ω_{01} from one run to the next.

The tendency of noise power to grow at low frequency means that in many settings the dephasing rate is expected to be much larger than the relaxation rate. In fact, the physical process responsible for dephasing may have so little power at frequency ω_{01} that a completely different process dominates relaxation. For example, in an ion trap dephasing may arise from voltage fluctuations in the trapping electrodes while relaxation is due to spontaneous photon emission. For electron spins in quantum dots dephasing arises from interactions with a bath of many nuclear spins, while relaxation is due to spontaneous phonon emission. (To exchange energy ω_{01} with the bath, many nuclear spins must flip, a highly suppressed process.)

Spin echo

One way to tame the damaging effects of low frequency noise is to invoke the spin echo trick. For example, when observing the dephasing of a spin evolving for time T, we may apply a fast pulse that flips the spin about the x-axis at time $T/2$. Then the effects of low-frequency phase noise during the second half of the evolution will tend to compensate for the effects of the phase noise during the first half.

If we use this trick, the damping factor applied to $|0\rangle\langle 1|$ is again given by

$$\exp\left(-\frac{1}{2}\int_{-\infty}^{\infty}\frac{dw}{2\pi}\tilde{K}(\omega)W_T(\omega)\right) \tag{17}$$

but with a modified window function

$$W_T(\omega) = \left|\int_0^T dt J(t) e^{i\omega t}\right|^2 , \tag{18}$$

where $J(t)$ is a modulating function that expresses the effect of the spin echo pulse sequence. For example, if we flip the spin at time $T/2$, then $J(t)$ is $+1$ in the interval $[0, T/2]$ and -1 in the interval $[T/2, T]$, and therefore

$$\begin{aligned} W_T(w) \quad &= \frac{1}{\omega^2}\left|1 - 2e^{i\omega T/2} + e^{i\omega T}\right|^2 = \frac{1}{\omega^2}\left|\frac{1 - e^{i\omega T/2}}{1 + e^{i\omega T/2}}(1 + e^{i\omega T/2})(1 - e^{i\omega T/2})\right|^2 \\ &= \tan^2(\omega T/4) \cdot \frac{4}{\omega^2}\sin^2(\omega t/2) \ . \end{aligned} \tag{19}$$

In effect, the spin echo modifies $\tilde{K}(\omega)$ by the multiplicative factor $\tan^2(\omega T/4)$, which suppresses the low frequency noise.

If the spin echo sequence consists of $2N-1$ equally spaced spin flips, then the window function becomes

$$W = \left|1 - 2x + 2x^2 - 2x^3 + \cdots - 2x^{2N-1} + x^{2N}\right|^2 , \tag{20}$$

where $x = e^{i\omega T/2N}$, or

$$W_T(\omega) = \left|\frac{1-x}{1+x}(1-x^{2N})\right|^2 = \tan^2(\omega T/4N) \cdot \frac{4}{\omega^2}\sin^2(\omega t/2) \ . \tag{21}$$

The peak of the frequency window gets shifted to the vicinity of $\omega = 2\pi N/T$.

Here we have described how spin echos can protect a stored qubit from low frequency noise. If we wish to perform a quantum gate with a nontrivial action on the qubit, it is possible to design a composite pulse sequence that realizes the gate while also suppressing the low frequency noise. Such procedures tend to flatten the noise power spectrum, and therefore make the noise more nearly symmetric (by reducing the disparity between the dephasing rate and the relaxation rate).

Spin-boson model

The spin-boson model is a more refined model of phase noise, in which the stochastic classical field $f(t)$ is replaced by a quantized bath of harmonic oscillators. Actually, the spin-boson model is said (e.g., by Leggett) to provide a reasonable description of the decoherence of an oscillating system rather generally, at least for the case where the system is weakly coupled to each of many degrees of freedom in the environment.

In this model, the Hamiltonian for the bath and for the coupling of the bath to the system is

$$H_B + H_{SB} = \sum_k \omega_k a_k^\dagger a_k - \frac{1}{2}\sigma_z\left(\sum_k g_k a_k + g_k^* a_k^\dagger\right) \tag{22}$$

There are many oscillators, so that the sum over k can be approximated by a frequency integral:

$$\sum_k |g_k|^2 \approx \int_0^\infty d\omega J(\omega) \ , \tag{23}$$

where $J(\omega)$ is the spectral function of the oscillator bath. If the Hamiltonian H_S of the system commutes with H_{SB} (for example if $H_S = \frac{1}{2}\omega_{01}\sigma_z$), then we can solve the model in closed form. For more general H_S, a formal solution can be written down, but the model is tractable only if appropriate approximations are made (for example, the "non-interacting blip approximation — NIBA — introduced by Leggett et al. for the study of macroscopic quantum coherence).

Let us assume that the bath is in thermal equilibrium at temperature β^{-1}. In principle, the coupling to the system could tweak the equilibrium distribution of the bath, but we assume that this effect is negligible, because the bath is much bigger than the system. Thus the fluctuations of the bath are Gaussian, and the average over the ensemble of classical functions in our previous analysis can be replaced by the thermal expectation value:

$$[f(t)f(0)]_f \mapsto \langle f(t)f(0)\rangle_\beta \equiv \mathrm{tr}\left(e^{-\beta H} f(t)f(0)\right) \ , \tag{24}$$

where now $f(t)$ denotes the operator

$$f(t) = e^{itH_B} f(0) e^{-itH_B} = \sum_k \left(g_k a_k e^{-i\omega_k t} + g_k^* a_k^\dagger e^{i\omega_k t}\right) \ . \tag{25}$$

We see that

$$K_\beta(t) \equiv \langle f(t)f(0)\rangle_\beta = \sum_k |g_k|^2 \langle e^{-i\omega_k t} a_k a_k^\dagger + e^{i\omega_k t} a_k^\dagger a_k \rangle_\beta \ . \tag{26}$$

From the Planck distribution,

$$\begin{aligned} \langle a_k^\dagger a_k\rangle_\beta &= \frac{1}{e^{\beta\omega}-1} = \frac{1}{2}\coth(\beta\omega_k/2) - \frac{1}{2} \\ \langle a_k a_k^\dagger\rangle_\beta &= \langle a_k^\dagger a_k + 1\rangle_\beta = \frac{1}{2}\coth(\beta\omega_k/2) + \frac{1}{2}. \end{aligned} \tag{27}$$

Fourier transforming, we find the spectral density of the noise

$$\tilde{K}_\beta(\omega) \equiv \int_{-\infty}^{\infty} dt\ e^{i\omega t} K_\beta(t) = \sum_k |g_k|^2 \left(2\pi\delta(\omega-\omega_k)\langle a_k a_k^\dagger\rangle_\beta + 2\pi\delta(\omega+\omega_k)\langle a_k^\dagger a_k\rangle_\beta\right) \tag{28}$$

that is,

$$\begin{aligned} \tilde{K}_\beta(\omega) &= \pi J(\omega)\left(\coth(\beta\omega/2)+1\right) \ , \quad \omega > 0 \ , \\ \tilde{K}_\beta(\omega) &= \pi J(\omega)\left(\coth(\beta\omega/2)-1\right) \ , \quad \omega < 0 \ . \end{aligned} \tag{29}$$

Thus the spectral density $\tilde{K}_\beta(\omega)$ of the noise at positive frequency ω is enhanced relative to the spectral density $\tilde{K}_\beta(-\omega)$ at negative frequency by the Boltzmann factor $e^{\beta\omega}$ (the "KMS condition"). This spectral asymmetry ensures that the rate for spontaneous decay of the system is enhanced relative to the thermal excitation rate by the Boltzmann factor, enforcing detailed balance of the decay and excitation rates in thermal equilibrium. (See eq.(70) and (71) below.)

Since the window function $W_T(\omega)$ is an even function of ω, only the even part of $\tilde{K}_\beta(\omega)$ contributes to the attenuation of $|0\rangle\langle 1|$; the attenuation factor is

$$\exp\left(-\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tilde{K}_\beta(\omega)W_T(\omega)\right) \ , \tag{30}$$

which therefore becomes

$$\exp\left(-\int_0^{\infty} d\omega J(\omega)\frac{2\sin^2(\omega T/2)}{\omega^2}\coth(\beta\omega/2)\right) \ . \tag{31}$$

A dephasing *rate* can be identified if the spectral function $J(\omega)$ behaves suitably at low frequency; the attenuation factor is $e^{-\Gamma_2 T}$ in the limit $T\to\infty$ where

$$\Gamma_2 = \lim_{\omega\to 0}\tilde{K}_\beta(\omega) = \lim_{\omega\to 0} 2\pi J(\omega)/(\beta\omega) \ , \tag{32}$$

assuming that this limit exists. That is, there is a dephasing rate $\Gamma_2 = 2\pi A\beta^{-1}$ (for $T \gg \beta$) provided that $J(\omega)\approx A\omega$ at low frequency, the "Ohmic" case.

At strictly zero temperature, we have $\coth(\beta\omega/2) = 1$. Then in the extreme "sub-Ohmic" case where $J(\omega)\approx A$ (a nonzero constant at low frequency), the dephasing rate is $\Gamma_2 = A\int_0^\infty(\sin^2 x)/x^2 = \frac{\pi}{2}A$. In the Ohmic case $J(\omega)\approx A\omega$, rather than a dephasing rate we find that our expression for the attenuation factor diverges logarithmically in the ultraviolet; it is $\exp\left(-A\log(\omega_c T)\right)$, where ω_c denotes a high-frequency cutoff. Thus the off-diagonal terms in the density operator decay like a power of $1/T$, rather than exponentially in T.

Multi-qubit decoherence

Now we generalize the spin-boson model to describe n qubits interacting collectively with a common oscillator bath:

$$H_{SB} = -\sum_l \frac{1}{2}\sigma_z^l \left(\sum_k g_k^l a_k + g_k^{l*} a_k^\dagger\right) \ . \tag{33}$$

The coupling parameter g_k^l quantifies the strength of the interaction of the kth oscillator with the lth spin. In this version of the model, since a single oscillator may couple to many spins, the fluctuations of that one oscillator can contribute to the decoherence of many spins. Thus the model incorporates a mechanism for collective decoherence.

Again, since we assume that the action of each oscillator on each spin is diagonal in the z basis, the spins will decohere in this basis. Let $|\eta\rangle = |\eta_{n-1}\eta_{n-2}\ldots\eta_1\eta_0\rangle$ denote an n-qubit state expressed in this basis. Then in time T, the density matrix element $|\eta\rangle\langle\mu|$ evolves to

$$\begin{aligned}\left\langle e^{-i\int_0^T H_{SB}}|\eta\rangle\langle\mu|e^{i\int_0^T H_{SB}}\right\rangle_\beta &= |\eta\rangle\langle\mu| \left\langle \exp\left(i\sum_l \left(\eta_l - \mu_l\right)\int_0^T dt f^l(t)\right)\right\rangle_\beta \\ = |\eta\rangle\langle\mu| \exp\left(-\frac{1}{2}\sum_{l,m}\left(\eta_l-\mu_l\right)\left(\eta_m-\mu_m\right)\int_0^T dt\int_0^T dt'\left\langle f^l(t) f^m(t')\right\rangle_\beta\right) \ , \end{aligned}\tag{34}$$

where

$$f^l(t) = \sum_k \left(g_k^l a_k e^{-i\omega_k t} + g_k^{l*} a_k^\dagger e^{i\omega_k t}\right) \ , \tag{35}$$

and therefore

$$\tilde{K}_\beta^{lm}(t-t') \equiv \left\langle f^l(t) f^m(t')\right\rangle_\beta = \sum_k g_k^l g_k^{m*} e^{-i\omega_k(t-t')}\langle a_k a_k^\dagger\rangle_\beta + g_k^{l*} g_k^m e^{i\omega_k(t-t')}\langle a_k^\dagger a_k\rangle_\beta \ . \tag{36}$$

Thus the factor by which $|\eta\rangle\langle\mu|$ is attenuated is

$$\exp\left(-\sum_{l,m} C_{lm}(\eta_l-\mu_l)(\eta_m-\mu_m)\right) \ , \tag{37}$$

where

$$C_{lm} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tilde{K}_\beta^{lm}(\omega) W_T(\omega) \ , \tag{38}$$

and

$$\tilde{K}_\beta^{lm}(\omega) = \pi\sum_k \left(\delta(\omega-\omega_k) g_k^l g_k^{m*}(\coth(\beta\omega_k/2)+1) + \delta(\omega+\omega_k) g_k^{l*} g_k^m(\coth(\beta\omega_k/2)-1)\right) \ . \tag{39}$$

A natural assumption is that k labels the momentum of a mode of a quantized field, e.g. that the qubits are immersed in a bath of thermal photons or phonons, so that $g_k^l = g_k e^{ik\cdot r_l}$, or

$$H_{SB} = \sum_l \frac{1}{2}\sigma_z^l\left(\sum_k g_k a_k e^{ik\cdot r_l} + g_k^* a_k^\dagger e^{-ik\cdot r_l}\right) \ ; \tag{40}$$

then

$$\tilde{K}_\beta^{lm}(\omega) = \quad \pi \sum_k |g_k|^2 \left(\delta(\omega-\omega_k)e^{ik\cdot(r_l-r_m)}(\coth(\beta\omega_k/2)+1)\right.$$
$$\left. +\delta(\omega+\omega_k)e^{-ik\cdot(r_l-r_m)}(\coth(\beta\omega_k/2)-1)\right) \ . \tag{41}$$

To go further, we need to know more about how k is related to ω_k. So let us assume the linear dispersion relation $\omega_k = |k|$, and an isotropic three-dimensional density of states. We can express $\tilde{K}_\beta(\omega)$ in terms of the bath spectral function $J(\omega)$ by averaging over the values of k that correspond to a fixed value of ω_k:

$$\frac{1}{2}\int_{-1}^{1} d\cos\theta e^{i\omega r\cos\theta} = \frac{\sin\omega r}{\omega r} \ , \tag{42}$$

thus obtaining

$$C_{lm} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tilde{K}_\beta^{lm}(\omega)W_T(\omega) = \int_0^\infty d\omega J(\omega)\frac{2\sin^2(\omega T/2)}{\omega^2}\coth(\beta\omega/2)\frac{\sin\omega|r_l-r_m|}{\omega|r_l-r_m|} \ . \tag{43}$$

This formula agrees with eq.(2) in Klesse and Frank (quant-ph/0505153), except that their Hamiltonian differs from mine by a factor of 2, so their formula seems to be missing a factor of 4. (Note that I am using T here to denote time, and β^{-1} to denote temperature.)

For modes with wavelength small compared to the separation between qubits, the oscillations of $\sin\omega r$ suppress the off-diagonal terms in the matrix C_{lm}, and the attenuation factor becomes

$$\exp\left(-\sum_l C_{ll}(\eta_l-\mu_l)^2\right) = \prod_l e^{-C(\eta_l\oplus\mu_l)} \ ; \tag{44}$$

the n qubits all decohere independently in the manner described by the single-qubit spin-boson model. But the thermal fluctuations of the long-wavelength modes induce collective decoherence. In the extreme case $|k|r \ll 1$ for all pairs of qubits, C_{lm} is independent of r, and we have instead

$$\exp\left(-C\sum_l(\eta_l-\mu_l)\sum_m(\eta_m-\mu_m)\right) = \exp\left(-C(|\eta|-|\mu|)^2\right) \ , \tag{45}$$

where $|\eta|$ denotes the Hamming weight of the binary string η. Thus in the case of extreme collective interference, the attenuation of $|\eta\rangle\langle\mu|$ is determined by the difference in the *total* σ_z of the states $|\eta\rangle$ and $|\mu\rangle$, since the low-frequency oscillators couple to the total spin. In contrast, the high-frequency oscillators "look at" the spins one at a time and the strength of the attenuation is determined by the number of spins for which there is a mismatch between η_l and μ_l. In both cases the constant C is given by the $|r_l - r_m| \to 0$ limit of eq.(43).

In effect, then, the high-frequency component of the bath behaves like independent baths interacting with each qubit. In that case, for fixed time T and at sufficiently weak coupling, the probability that t qubits are afflicted by Z errors arising from the fluctuations of the bath is

$$P_t = p^t(1-p)^{n-t} \ , \quad p = \frac{1}{2}\left(1-e^{-C}\right) \ . \tag{46}$$

But as Klesse and Frank correctly point out, in the limit of collective decoherence, and even for $p \ll 1$, Z errors can act on a fixed fraction much larger than p of the n qubits, with a nonzero probability independent of n in the limit $n \to \infty$. The fluctuations of the low frequency modes apply correlated kicks to all n qubits. Though these kicks are typically small when the coupling is weak, rare large fluctuations can damage many qubits at once.

However, Klesse and Frank consider an unreasonable limit, in which the maximal spatial separation of a pair of qubits is held fixed as the number of qubits n increases. We should consider how C_{lm} behaves when the separation $|r_l - r_m|$ is large compared to other natural scales. In this case we may presume that the integral over the spectrum of the bath will be dominated by small values of ω such that $\beta\omega \ll 1$ and $\omega T \ll 1$. Then we can approximate eq.(43) by

$$C(r) \approx \int_0^\infty d\omega J(\omega) \cdot \frac{1}{2}T^2 \cdot \frac{2}{\beta\omega} \cdot \frac{\sin \omega r}{\omega r} \ , \tag{47}$$

and in the Ohmic case $J(\omega) \approx A\omega$ we find

$$C(r) \approx A\beta^{-1}T^2 \int_0^\infty d\omega \frac{\sin \omega r}{\omega r} = A\beta^{-1}r^{-1}T^2 \int_0^\infty dx \frac{\sin x}{x} = \frac{\pi A \beta^{-1}}{2r} T^2 \ . \tag{48}$$

If the distance r scales like a power of n, then the collective decoherence should be weak enough for quantum error correction to work effectively; still it would be useful to study this issue in more detail. How well does the spin-boson model assumed by Klesse and Frank, in which the long-wavelength modes of the bath coupled collectively to many qubits, apply to realistic devices? Also note that the effects of low-frequency fluctuations can be further suppressed through the use of spin-echo pulse sequences, and that collective noise can be suppressed by encoding logical qubits in "decoherence-free subspaces."

Indeed, the model of collective decoherence discussed by Klesse and Frank can be equivalently described as a time-fluctuating but spatially uniform classical magnetic field that couples to the $\hat{z}$ component of the total spin of the qubits. If we encode logical qubits in two-qubit blocks according to

$$|\bar{0}\rangle = |01\rangle \ , \quad |\bar{1}\rangle = |10\rangle \ , \tag{49}$$

then all codewords have the same total σ_z (namely 0), so that the magnetic field has no effect at all. Nevertheless, it is sometimes useful to keep the Klesse-Frank model in mind, as it cautions us that we cannot expect to prove that fault-tolerant quantum computing succeeds without appropriate assumptions about the oscillator bath and/or about the encoding used.

Interaction picture

When H_S does not commute with H_{SB} (including the case where the system Hamiltonian is time dependent), we can describe the joint evolution of the system and bath perturbatively, using the interaction picture. To find the time evolution operator $U(T,0)$ determined by the Hamiltonian

$$H = H_S + H_B + H_{SB} \ , \tag{50}$$

we divide the time interval $[0, T]$ into many infinitesimal intervals, each of width Δ, for which the evolution operator is

$$U(t+\Delta, t) = e^{-i\Delta H} = e^{-i\Delta H_S(t)} e^{-i\Delta H_B(t)} \left(I - i\Delta H_{SB}(t)\right) \tag{51}$$

The mth order term in perturbation theory in H_{SB} can be expressed as

$$(-i)^m \sum_{t_1, t_2, \ldots, t_m} U_0(T, t_m) H_{SB}(t_m) U_0(t_m, t_{m-1}) H_{SB}(t_{m-1}) \ldots H_{SB}(t_1) U_0(t_1, 0) \ , \tag{52}$$

where U_0 is the time evolution operator for Hamiltonian $H_S + H_B$, and $t_m > t_{m-1} > \cdots > t_2 > t_1$. That is, the perturbation H_{SB} couples the system to the bath at m distinct times, and in between these events, the system and bath evolve independently.

If we define the interaction picture by $A_I(t) = U_0(0,t) A(t) U_0(t, 0)$, we can express the joint evolution operator as

$$\begin{aligned} U(T,0) \qquad &= U_0(T,0) \cdot \left(\sum_{m=0}^{\infty} (-i)^m \sum_{t_1,t_2,\ldots,t_m} H_{I,SB}(t_m) H_{I,SB}(t_{m-1}) \ldots H_{I,SB}(t_1) \right) \\ &= U_0(T,0) \cdot T \exp\left(-i \int_0^T dt H_{I,SB}(t) \right) \ , \end{aligned} \tag{53}$$

or in other words,

$$U_I(T,0) \equiv U_0(0,T) U(T,0) = T \exp\left(-i \int_0^T dt H_{I,SB}(t) \right) \ , \tag{54}$$

where $T\exp$ denotes the time-ordered exponential.

Without loss of generality, we may express the system-bath coupling as $H_{SB} = A_i \otimes B_i$ (with the understanding that the repeated index i is summed), where A_i acts on the system and B_i acts on the bath (we may also assume that each A_i and each B_i is Hermitian). Suppose that, as in our discussion of the spin-boson model, the correlations of the bath operators $\{B_i\}$ are Gaussian — as would be the case for example if the state ρ_B of the bath were a thermal state of noninteracting harmonic oscillators. The two-point correlation functions ("propagators") in the bath may be denoted

$$\begin{aligned} &\langle B_i(t_2) B_j(t_1) \rangle = \mathrm{tr}_B \left(B_i(t_2) B_j(t_1) \rho_B \right) \\ &= \mathrm{tr}_B \left(U_B(T,t_2) B_i U_B(t_2,t_1) B_j U_B(t_1,0) \rho_B U_B(0,T) \right) = K_{ij}(t_2 - t_1) \ ; \end{aligned} \tag{55}$$

the interaction picture system density operator $\rho_{I,S} \equiv U_S(0,t)\rho_S(t)U_S(t,0)$ evolves according to

$$\begin{aligned} &\rho_{I,S}(T) = \sum_{m,n=0}^{\infty} \frac{(-i)^m (i)^n}{n!\ m!} \int_0^T dt_1 dt_2 \ldots dt_m \int_0^T ds_1 ds_2 \ldots ds_n \\ &T\Big(A_{i_m}(t_m) \ldots A_{i_2}(t_2) A_{i_1}(t_1) \Big) \rho_S(0) \bar{T}\Big(A_{j_1}(s_1) A_{j_2}(s_2) \ldots A_{j_n}(s_n) \Big) \\ &\Big\langle B_{j_1}(s_1) B_{j_2}(s_2) \ldots B_{j_n}(s_n) B_{i_m}(t_m) \ldots B_{i_2}(t_2) B_{i_1}(t_1) \Big\rangle \ . \end{aligned} \tag{56}$$

(Here $\bar{T}$ denotes *anti*-time ordering, and $A_i(t) = U_S(0,t)A_iU_S(t,0)$.) Note that we use the unperturbed system dynamics to define the interaction picture for system operators, and the unperturbed bath dynamics to define the interaction picture for the bath operators. In principle these operators could have intrinsic time dependence, aside from the time dependence arising from transforming to the interaction picture; the bath correlators will be assumed to be Gaussian, but need not be stationary.) The (Gaussian) expectation value of the string of bath operators can be expressed as a sum over all possible "contractions" of pairs of operators. Thus we obtain a sum of diagrams, where in each diagram the points where the perturbations act on the system are connected pairwise by bath propagators. Note that there are three types of contractions – we can contract two t's, two s's, or an s and a t.

This expansion gives the evolution exactly, at least under the assumption that the correlations of the bath are Gaussian. By making a further approximation, we can obtain a fairly simple integro-differential equation for $\rho_{I,S}(T)$. This approximation, called the "Born" approximation, is justified if the coupling of the system to the bath is sufficiently weak *and* the correlation time of the bath is sufficiently short. In diagrammatic terms, the approximation is to include in the sum only the diagrams for which no two contractions "cross one another." That is, if a vertex at time t_1 is contracted with a vertex at time $t_2 > t_1$, then there are no vertices in the time interval $[t_1, t_2]$ that are also contracted with other vertices. And similarly, if s is contracted with $t > s$, then there are no vertices in the time interval $[s, t]$ that are contracted with other vertices. Physically, in the Born approximation we are assuming that the system evolves slowly compared to the correlation time of the bath.

Now if we once differentiate the expression for $\rho_{I,S}(T)$ in eq.(56) with respect to T, and we assume that only the diagrams that are allowed under the Born approximation are included, we obtain terms with one of the system operators acting at time T from the left or from the right. The corresponding bath operator at time T is contracted with a bath operator at an earlier time. Apart from the bath propagator, what remains is the diagram sum that generates $\rho_{I,S}$ at the earlier time. That is, we obtain the integro-differential equation:

$$\begin{aligned}\dot{\rho}_{I,S}(T) = \quad & \int_0^T dt \Big(\big\langle B_j(t)B_i(T)\big\rangle A_i(T)\rho_{I,S}(t)A_j(t) + \big\langle B_j(T)B_i(t)\big\rangle A_i(t)\rho_{I,S}(t)A_j(T) \\ & - \big\langle B_j(T)B_i(t)\big\rangle A_j(T)A_i(t)\rho_{I,S}(t) - \big\langle B_j(t)B_i(T)\big\rangle \rho_{I,S}(t)A_j(t)A_i(T)\Big) \ . \qquad (57)\end{aligned}$$

Note that we can easily check that $\mathrm{tr}\,[\dot{\rho}_{I,S}(T)] = 0$ and that $\dot{\rho}_{I,S}$ is Hermitian, as is required for the normalization and Hermiticity of the density operator to be preserved. In the Born approximation (or "first Born approximation") we retain on the right-hand side of eq.(57) the leading (quadratic) term in an expansion in powers of A. In the "second Born approximation" we would expand the right-hand side to quartic order in A, and so on. Thus the Born approximation is the first term in a systematic weak-coupling expansion; however I emphasize again that the bath correlation function must be short enough for the Born approximation to be well justified — only then does it make sense to include the diagrams without crossings that arise when eq.(57) is integrated, while excluding diagrams with crossings that begin to appear only in the next order of the weak-coupling expansion.

A stronger assumption is that the time-dependence of $\rho_{I,S}(t)$ can be completely ignored on the right-hand side — then we may replace $\rho_{I,S}(t)$ inside the integral by $\rho_{I,S}(T)$, obtaining a first-order differential equation for $\rho_{I,S}(T)$. This is called the "Born-Markov" approximation.

In what might be called the "extreme Markovian limit," the correlation time of the bath is actually zero: $\langle B_j(t)B_i(t')\rangle = K_{ji}\delta(t-t')$, and we obtain the Lindblad master equation in the interaction picture:

$$\dot\rho_{I,S}(T) = K_{ji}\Big(2A_i(T)\rho_{I,S}(T)A_j(T) - A_j(T)A_i(T)\rho_{I,S}(T) - \rho_{I,S}(T)A_j(T)A_i(T)\Big)\ . \quad (58)$$

Radiative damping

In the spin boson model, the oscillator bath couples to σ_z. Thus the qubit decoheres in the σ_z eigenbasis, but there is no relaxation or excitation, i.e., the excited state of the qubit does not spontaneously decay to the ground state, and the ground state does not become thermally excited.

Another interesting model of decoherence is the radiatively damped two-level atom. Here the Hamiltonian of system and bath is

$$H = \frac{1}{2}\omega\sigma_z + \sum_k \omega_k a_k^\dagger \omega_k + \sum_k \left(g_k a_k \sigma^+ + g_k^* a_k^\dagger \sigma^-\right)\ ; \quad (59)$$

the atom becomes excited if it absorbs energy from the radiation bath, and it can decay by emitting a radiation quantum. If the radiation bath is thermal, then the $\langle a(t)a(0)\rangle_\beta$ and $\langle a^\dagger(t)a^\dagger(0)\rangle_\beta$ correlators vanish, so that the Born-approximation equation of motion in the interaction picture becomes

$$\begin{aligned}\dot\rho(T) = \int_0^T dt \sum_k |g_k|^2\Big[& \langle a_k(T)a_k^\dagger(t)\rangle_\beta \left(\sigma^-(t)\rho(t)\sigma^+(T) - \sigma^+(T)\sigma^-(t)\rho(t)\right)\\ + & \langle a_k(t)a_k^\dagger(T)\rangle_\beta \left(\sigma^-(T)\rho(t)\sigma^+(t) - \rho(t)\sigma^+(t)\sigma^-(T)\right)\\ + & \langle a_k^\dagger(T)a_k(t)\rangle_\beta \left(\sigma^+(t)\rho(t)\sigma^-(T) - \sigma^-(T)\sigma^+(t)\rho(t)\right)\\ + & \langle a_k^\dagger(t)a_k(T)\rangle_\beta \left(\sigma^+(T)\rho(t)\sigma^-(t) - \rho(t)\sigma^-(t)\sigma^+(T)\right)\Big]\ . \end{aligned} \quad (60)$$

It is now understood that $\rho(t)$ and $\sigma^\pm(t)$ denote operators in the H_S (i.e., the system's) interaction picture, while $a_k(t)$, $a(t)_k^\dagger$ are operators in the H_B (i.e., the bath's) interaction picture. (To write down all the terms correctly in eq.(60), one recalls that e.g. $\sigma^+(t)$ acting on the atom is always accompanied by $a(t)$ acting on the bath, while $\sigma^-(t)$ is accompanied by $a^\dagger(t)$. One also notes that in a term of the form $\sigma\sigma\rho$, $\sigma(T)$ must by furthest to the *left*, while in a term of the form $\rho\sigma\sigma$, $\sigma(T)$ must be furthest to the *right*.)

Defining the positive and negative frequency parts of the thermal correlators by

$$\begin{aligned} K_\beta^+(t) &= \sum_k |g_k|^2 e^{-i\omega_k t}\langle a_k a_k^\dagger\rangle_\beta\ ,\\ K_\beta^-(t) &= \sum_k |g_k|^2 e^{i\omega_k t}\langle a_k^\dagger a_k\rangle_\beta\ , \end{aligned} \quad (61)$$

and using $\sigma^+(t) = e^{i\omega t}\sigma^+$, $\sigma^-(t) = e^{-i\omega t}\sigma^-$, we may write

$$\dot\rho(T) = \int_0^T dt\Big[\quad K_\beta^+(T-t)e^{i\omega(T-t)}\left(\sigma^-\rho(t)\sigma^+ - \sigma^+\sigma^-\rho(t)\right)$$

$$
\begin{aligned}
+ \quad & K_\beta^+(t-T)e^{-i\omega(T-t)}\left(\sigma^-\rho(t)\sigma^+ - \rho(t)\sigma^+\sigma^-\right) \\
+ \quad & K_\beta^-(T-t)e^{-i\omega(T-t)}\left(\sigma^+\rho(t)\sigma^- - \sigma^-\sigma^+\rho(t)\right) \\
+ \quad & K_\beta^-(t-T)e^{i\omega(T-t)}\left(\sigma^+\rho(t)\sigma^- - \rho(t)\sigma^-\sigma^+\right)\Big] \ . \qquad (62)
\end{aligned}
$$

In the Markovian limit, we regard the correlation time of the bath to be short compared to the time scale for the evolution of the system (in the interaction picture). Then we may extend the integral over t indefinitely into the past (replace the lower limit of integration by $-\infty$), and replace $\rho(t)$ by $\rho(T)$. Making the change of integration variable $s = T - t$, and noting that $K_\beta^+(t)^* = K_\beta^+(-t)$ and that $K_\beta^-(t)^* = K_\beta^-(-t)$, we define real quantities $\gamma, \delta, \kappa, \eta$ by

$$
\begin{aligned}
\int_0^\infty ds \ K_\beta^+(s)e^{i\omega s} &= \frac{1}{2}\gamma + i\delta \ , \\
\int_0^\infty ds \ K_\beta^+(-s)e^{-i\omega s} &= \frac{1}{2}\gamma - i\delta \ , \\
\int_0^\infty ds \ K_\beta^-(s)e^{-i\omega s} &= \frac{1}{2}\kappa + i\eta \ , \\
\int_0^\infty ds \ K_\beta^-(-s)e^{i\omega s} &= \frac{1}{2}\kappa - i\eta \ . \qquad (63)
\end{aligned}
$$

Making these substitutions, we find the interaction picture equation of motion

$$
\begin{aligned}
\dot\rho = \quad & \gamma\left(\sigma^-\rho\sigma^+ - \frac{1}{2}\sigma^+\sigma^-\rho - \frac{1}{2}\rho\sigma^+\sigma^-\right) \\
& +\kappa\left(\sigma^+\rho\sigma^- - \frac{1}{2}\sigma^-\sigma^+\rho - \frac{1}{2}\rho\sigma^-\sigma^+\right) \\
& -i\delta[\sigma^+\sigma^-, \rho] - i\eta[\sigma^-\sigma^+, \rho] \ . \qquad (64)
\end{aligned}
$$

Thus we find the expected Markovian Lindblad terms describing atomic decay with rate γ and excitation with rate κ. In addition, the fluctuations of the bath renormalize the system Hamiltonian. Noting that $\sigma^+\sigma^- = \frac{1}{2}(1+\sigma_z)$ and $\sigma^-\sigma^+ = \frac{1}{2}(1-\sigma_z)$ are projection operators onto the σ_z eigenstates with eigenvalues $+1$ and -1 respectively, we see that δ is an additive renormalization of the energy of the atomic excited state $|e\rangle$ ($\sigma_z = 1$), and η is an additive renormalization of the energy of the atomic ground state $|g\rangle$ ($\sigma_z = -1$).

At strictly zero temperature ($\beta = \infty$), $K_\infty^-(t) = 0$, so that κ and η vanish, while γ and δ remain nonzero. In the zero-temperature limit, the excited state can still decay, but there are no thermal quanta to drive the atom from the ground state to the excited state. There is also a surviving contribution to the renormalization of the energy splitting, the *Lamb shift*.

At zero temperature, $\langle a_k a_k^\dagger\rangle = 1$; therefore, in terms of the spectral function $J(\omega)$ of the bath, we may express $K_\beta^+(s)$ as

$$
K_\infty^+(s) = \int_0^\infty d\omega' J(\omega')e^{-i\omega' s} \ , \qquad (65)
$$

and so

$$
\frac{1}{2}\gamma_0 + i\delta_0 = \int_0^\infty ds K_\infty^+(s) = \int_0^\infty d\omega' J(\omega')\left(\lim_{T\to\infty}\int_0^T ds \ e^{i(\omega-\omega')s}\right) \ . \qquad (66)
$$

One way to make sense of the limit is to imagine that $\omega - \omega'$ has a small positive imaginary part, in which case we have

$$\frac{1}{2}\gamma_0 + i\delta_0 = \int_0^\infty d\omega' \frac{-i}{\omega' - \omega - i\epsilon} J(\omega') \ , \tag{67}$$

where it is understood that we are to take the limit $\epsilon \to 0^+$. Appearing in the integral is the Green function $\left(i\frac{d}{dt} - \omega\right)^{-1}$ for the time-dependent Schrödinger equation, with the pole at $\omega' = \omega$ infinitesimally displace above the real axis in the ω' plane (corresponding to a *advanced* boundary condition for the Green function, which is appropriate because we are to integrate over times *earlier* than T when we evaluate $\dot{\rho}_I(T)$). Also appearing in the integral is the bath's spectral function, essentially a density-of-states factor modulated by the strength of the coupling of the atom to the oscillators. The integral describes a virtual process at zero temperature, in which the atom makes a transition from the excited state to the ground state, emitting a quantum which is later absorbed as the atom returns to the excited state.

To find the real and imaginary parts $\frac{1}{2}\gamma + i\delta$ of the integral, we note that

$$\frac{-i}{\omega' - \omega - i\epsilon} = \pi\delta(\omega' - \omega) + PV\frac{-i}{\omega' - \omega} \ , \tag{68}$$

where PV denotes the Cauchy principal value; thus

$$\gamma = 2\pi J(\omega) \ , \quad \delta = PV\int_0^\infty d\omega' J(\omega')\frac{-i}{\omega' - \omega} \ . \tag{69}$$

The result for the decay rate, involving the emission of real quanta with energy ω, is just what one expects from "Fermi's Golden Rule," — the matrix element of the perturbation and the appropriate density-of-final-states factor have been absorbed into the bath's spectral function $J(\omega)$. The Lamb shift involves virtual quanta of all frequencies, and is potentially divergent, depending on the high-frequency behavior of the spectral function $J(\omega)$ (an ultra-violet divergence can be handled as in the standard renormalization program for quantum electrodynamics).

It is actually easier to get the result for the decay rate γ, for any value of β, directly from eq.(63):

$$\gamma = \int_0^\infty ds K_\beta^+(s)e^{i\omega s} + \int_0^\infty ds K_\beta^+(-s)e^{-i\omega s} = \int_{-\infty}^\infty ds K_\beta^+(s)e^{i\omega s} = \tilde{K}_\beta^+(\omega) \ , \tag{70}$$

and similarly

$$\kappa = \int_0^\infty ds K_\beta^-(s)e^{-i\omega s} + \int_0^\infty ds K_\beta^-(-s)e^{i\omega s} = \int_{-\infty}^\infty ds K_\beta^-(s)e^{-i\omega s} = \tilde{K}_\beta^-(-\omega) \ . \tag{71}$$

Thus one says that the spontaneous decay rate is determined by the strength of the positive frequency (thermal and vacuum) noise at the transition frequency ω, while the excitation rate is determined by the strength of the negative frequency (thermal) noise at the transition frequency. Of course, this statement holds very generally, not just for the particular model of radiative damping we have discussed here.

The weak-coupling limit generalized

In our model of radiative damping, the operators σ^+ and σ^- acting on the system have "definite frequency" — that is, they alter the energy of the system by a definite amount. In general, if the system Hamiltonian is time-independent, operators acting on the system can be expanded in a basis $\{A_\omega\}$ of operators with definite frequency:

$$[H_S, A_\omega] = -\omega A_\omega \ ; \tag{72}$$

this basis is determined by diagonalizing the adjoint action of the system Hamiltonian. An operator A_ω with frequency $\omega > 0$ *lowers* the energy of the system by $\hbar\omega$; its adjoint $A_\omega^\dagger$ has frequency $-\omega$ and *raises* the system's energy. An operator of definite frequency evolves in the interaction picture according to

$$e^{iH_S t} A_\omega e^{-iH_S t} = e^{-i\omega t} A_\omega \tag{73}$$

(as can be verified by differentiating both sides with respect to t).

Let the system-bath coupling be $A \otimes B$, where A and B are both Hermitian; hence the system operator A can be expanded as

$$A = \sum_\omega A_\omega = \sum_\omega A_\omega^\dagger \ . \tag{74}$$

Assuming that the bath's two point correlation function $K(t-T) = \langle B(t)B(T)\rangle$ is stationary (time-translation invariant), plugging into eq.(57) yields

$$\begin{aligned}\dot\rho(T) = \sum_{\omega,\omega'} \int_0^T dt \quad & \Big[K(t-T) e^{-i\omega' T} e^{i\omega t} \left(A_{\omega'} \rho(t) A_\omega^\dagger - \rho(t) A_\omega^\dagger A_{\omega'} \right) \\ + \quad & K(T-t) e^{-i\omega t} e^{i\omega' T} \left(A_\omega \rho(t) A_{\omega'}^\dagger - A_{\omega'}^\dagger A_\omega \rho(t) \right) \Big] \ ,\end{aligned} \tag{75}$$

and making the substitution $t = T - s$ we find

$$\begin{aligned}\dot\rho(T) = \sum_{\omega,\omega'} \Big[\quad & e^{i(\omega-\omega')T} \int_0^T ds\ K(-s) e^{-i\omega s} \left(A_{\omega'} \rho(T-s) A_\omega^\dagger - \rho(T-s) A_\omega^\dagger A_{\omega'} \right) \\ + \quad & e^{-i(\omega-\omega')T} \int_0^T ds\ K(s) e^{i\omega s} \left(A_\omega \rho(T-s) A_{\omega'}^\dagger - A_{\omega'}^\dagger A_\omega \rho(T-s) \right) \Big] \ ,\end{aligned} \tag{76}$$

The terms with $\omega \neq \omega'$ oscillate rapidly if the time T of observation is long compared to the inverse frequency difference $(\omega - \omega')^{-1}$; the oscillations average to zero and so these terms can be neglected. Thus, if we consider the "coarse-grained" evolution over a time that is long compared to the time scale set by the relevant "Bohr frequencies" of the system, then only the terms with $\omega = \omega'$ survive.

For the surviving terms, as in our discussion of radiative damping, if we replace $\rho(T-s)$ by $\rho(T)$ and extend the upper limit of the s integral to infinity (these approximations are reasonable if the time T is long compared to the correlation time $\tau_{\rm corr}$ of the bath, and if the interaction-picture evolution of the system is slow compared to $\tau_{\rm corr}$), then we obtain

$$\begin{aligned}\dot\rho = \quad & \sum_{\omega>0} \tilde K(\omega) \left(A_\omega \rho A_\omega^\dagger - \frac{1}{2} A_\omega^\dagger A_\omega \rho - \frac{1}{2} \rho A_\omega^\dagger A_\omega \right) \\ + \quad & \sum_{\omega>0} \tilde K(-\omega) \left(A_\omega^\dagger \rho A_\omega - \frac{1}{2} A_\omega A_\omega^\dagger \rho - \frac{1}{2} \rho A_\omega A_\omega^\dagger \right) \ ,\end{aligned} \tag{77}$$

(ignoring the energy shifts, which can be absorbed into the system Hamiltonian). I have separated the terms into two sums to emphasize that the negative frequency fluctuations of the bath excite the system, while the positive frequency fluctuations are responsible for relaxation. This evolution equation is sometimes said to be the *Davies master equation.* If the bath is in a thermal state that obeys the KMS condition $\tilde{K}(-\omega) = e^{-\beta\omega}\tilde{K}(\omega)$, a thermal state $\rho = e^{-\beta H_S}$ of the system will be stationary if it evolves according to this equation.

If the initial density operator is diagonal in the energy eigenstate basis of the system, then evolution governed by the Davies master equation preserves this property. The terms contributing to $\dot{\rho}$ with rapidly oscillating phases (which we argued can be neglected) are off-diagonal in the system's energy eigenstate basis. In a sense the oscillating phase is an artifact of working in the interaction picture and would not be present if we re-expressed the time derivative of the density operator in the Schrödinger picture. We emphasize again that to justify ignoring the off-diagonal terms we must coarse grain in time. That is, it is implicit that we consider not the instantaneous value of $\dot{\rho}$ in the interaction picture, but rather the result of integrating $\dot{\rho}$ for a sufficiently long finite time, such that it is a good approximation to say that the oscillating phase averages to zero.

Alicki, Lidar, and Zanardi (quant-ph/0506201), invoking the derivation of the Davies master equation, have suggested that a Markovian noise model cannot be applicable over a time period comparable to the working period of a quantum gate. Their observation is that the time scale for realizing a gate is determined by a difference of two system frequencies: $\tau_{\rm gate}^{-1} \approx \omega - \omega'$. For the Markovian approximation to be good, then, the observation time T must be large compared to $\tau_{\rm gate}$.

This conclusion seems too strong. We should keep in mind that the system may have a range of relevant energy scales, just as the fluctuations of the bath may also span a range of frequencies. Thus the dominant noise frequencies might differ by orders of magnitude from the frequencies that characterize the speed of the gates.

Phase noise and quantum gates

Since phase noise tends to be more prevalent than relaxation in many potential realizations of quantum hardware, it would seem to make sense to use fault-tolerant protocols that protect gates more effectively against phase errors than against bit flips. Formulating such protocols faces interesting challenges, because we need to be careful to avoid using gates (such as Hadamard gates) that transform phase errors into bit flips.

But one should also ask whether the concept of a quantum computer that is subject to much stronger phase errors than bit flips really makes sense in a reasonable physical context. The trouble is that even if resting (memory) qubits are subject only to phase errors and no bit flips, the phase errors can be transformed into bit flips *during* the execution of the gate.

As an example, suppose we are trying to perform a NOT (σ_x) gate on a qubit that is in a fluctuating magnetic field that causes phase noise. We may take the Hamiltonian to be

$$H = \frac{1}{2}g(t)\sigma_x + \frac{1}{2}f(t)\sigma_z \ . \tag{78}$$

Here $g(t)$ is a deterministic classical function that can be controlled by the operator of the quantum computer, but $f(t)$ is fluctuating; the fluctuations are assumed to be Gaussian with mean zero and covariance $[f(t)f(t')]_f = K(t-t')$.

The time evolution operator for the ideal system is

$$U_S(t) = \exp\left(-i\frac{1}{2}G(t)\sigma_x\right) \ , \quad G(t) = \int_0^t dt \ g(t) \ . \tag{79}$$

Therefore, a NOT can be implemented in time T by choosing $g(t)$ such that $G(T) = \pi$ (a "π pulse").

We can analyze the effect of the noise on the ideal gate using interaction-picture perturbation theory, as in eq.(56). Let us work to lowest nontrivial order (that is, quadratic order) in the expansion, which is a reasonable approximation if the effect of the dephasing on the gate is small. In the case where the system-bath coupling is simply $H_{SB} = A \otimes B$ (A is a Hermitian operator acting on the system, and B is a Hermitian operator acting on the bath), then to quadratic order the evolution equation for the system's interaction picture density operator ρ_I is

$$\begin{aligned}\rho_I(T) = &\int_0^T dt \int_0^T ds \langle B(s)B(t)\rangle \ A(t)\rho_I(0)A(s) \\ &- \int_0^T ds \int_0^s dt \langle B(s)B(t)\rangle \ A(s)A(t)\rho_I(0) \\ &- \int_0^T dt \int_0^t ds \langle B(s)B(t)\rangle \ \rho_I(0)A(s)A(t)\end{aligned} \tag{80}$$

(for clarity the time ordering is indicated explicitly). In the Markovian limit, with $K(t-t') = \kappa\delta(t-t')$, this expression becomes

$$\rho_I(T) = \kappa \int_0^T dt \left(A(t)\rho_I(0)A(t) - \frac{1}{2}A(t)A(t)\rho_I(0) - \frac{1}{2}\rho_I(0)A(t)A(t)\right) \ . \tag{81}$$

Now, in the case of our model describing a NOT gate subject to dephasing,

$$A(t) = \frac{1}{2}U_S(t)^\dagger \sigma_z U_S(t) = \frac{1}{2}\left(\sigma_z \cos G(t) + \sigma_y \sin G(t)\right) \ , \tag{82}$$

so that in the Markovian case, we have

$$\rho_I(T) = \frac{\kappa}{4}\int_0^T dt \left[\sigma_z \cos G(t) + \sigma_y \sin G(t)\right] \rho_I(0) \left[\sigma_z \cos G(t) + \sigma_y \sin G(t)\right] + \dots \tag{83}$$

where the ellipsis represents the $\sigma\sigma\rho$ and $\rho\sigma\sigma$ terms needed to ensure that the evolution is normalization preserving. Expanding this expression we obtain

$$\begin{aligned}\rho_I(T) = &\left(\frac{\kappa}{4}\int_0^T dt \ \cos^2 G(t)\right)\sigma_z\rho_I(0)\sigma_z + \left(\frac{\kappa}{4}\int_0^T dt \ \sin^2 G(t)\right)\sigma_y\rho_I(0)\sigma_y \\ &+ \left(\frac{\kappa}{4}\int_0^T dt \ \cos G(t)\sin G(t)\right)\left(\sigma_z\rho_I(0)\sigma_y + \sigma_y\rho_I(0)\sigma_z\right) + \dots\end{aligned} \tag{84}$$

To do a NOT gate, we choose $G(0) = 0$ and $G(T) = \pi$. If the pulse is symmetric about $t = T/2$, then the third, off-diagonal, term is sure to vanish. The sum of the coefficients of the first two terms is $\kappa T/4$, but the relative size of the two terms depends on the pulse shape. In particular, if $G(t)$ is close to either 0 or π most of the time, except for a rapid swing from 0 to π, then the second term will be suppressed.

Of course, this is obvious. The effect of the first term is the same as if the gate were first applied ideally, followed by a σ_z error occurring with a specified probability, while the second term describes a σ_y error following the ideal gate. If the NOT gate is performed very rapidly, followed by a pause for time T, the phase error is like one that afflicts a resting memory qubit stored for time T. But if the NOT gate is performed more slowly, phase errors are more likely to occur while the qubit is rotating — the ideal system dynamics propagates such phase errors to bit flips. For example, if a σ_z error occurs right in the middle of the rotation by π in the yz-plane, then the spin will end up with values of σ_z and σ_x that are opposite to the values they would have if an ideal NOT gate had been applied — that is a σ_y error will have occurred.

Is there an alternative scheme for implementing gates that does not propagate phase errors to bit flips, other than the rapid pulse followed by a long pause? For example, if we can measure both σ_x and σ_z, then it is possible to realize universal quantum computation using only gates that can be implemented by turning on and off interaction Hamiltonians that commute with σ_z. But is it possible to realize a universal set of fault-tolerant encoded gates using such interactions, where the quantum code can correct many more phase errors than bit flips?

A new threshold estimate?

Aliferis-Gottesman-Preskill (AGP, quant-ph/0504218) proved a quantum threshold theorem that applies to non-Markovian noise, but unfortunately the AGP theorem does not apply when the noise is due to a bath of harmonic oscillators (as in the spin-boson model).

Let's recall the AGP argument. We may regard a quantum circuit as a unitary operator U_{SB} acting on the system qubits and the bath, which can be expressed as a *coherent* sum of "fault paths":

$$U_{SB} = \sum \text{fault paths} \ . \tag{85}$$

In each fault path, some of the gates are ideal, and some are designated as faulty. The defining feature of the *local noise* model considered by AGP is: Let $\mathcal{I}_r$ denote any particular set of r locations (i.e., gates) in the circuit, and let $E(\mathcal{I}_r)$ denote the sum of all terms in the fault path expansion such that all of the locations in $\mathcal{I}_r$ are faulty. Then

$$\|E((\mathcal{I}_r)\| \leq \eta^r \ , \tag{86}$$

where $\|\cdot\|$ denotes the operator (sup) norm. We say that η is the *strength* of the noise. AGP showed that if we apply "level reduction" to a fault-tolerant quantum circuit (reducing a circuit that uses a level-k concatenated quantum error-correcting code to an equivalent circuit using a level-($k{-}1$) code), then we obtain a new fault-path expansion which also obeys the local noise condition, with a new noise strength $\eta^{(1)} = O(\eta^2)$ that is smaller than η, provided $\eta < \eta_{\rm th}$. Applying level-reduction k times in succession proves the threshold theorem.

Furthermore, the local noise condition holds in a Hamiltonian framework. Suppose that U_{SB} arises from integrating the Schödinger equation for the time-dependent Hamiltonian

$$H = H_S + H_B + H_{SB} \; ; \tag{87}$$

Here H_S is the system Hamiltonian that realizes the ideal quantum circuit, H_B is an arbitrary Hamiltonian of the bath, and

$$H_{SB} = \sum_a H_{SB,a} \tag{88}$$

is a sum of terms where at each time each term corresponds to a particular location where a gate is being performed at that time. That is, if a denotes a single-qubit location, then $H_{SB,a}$ acts on only that system qubit (and has an arbitrary action on the bath); if a denotes a two-qubit location, then $H_{SB,a}$ acts on those two system qubits, etc. Then we may express the noise strength as

$$\eta = t_0 \cdot \text{Max} \; \|H_{SB,a}\| \; , \tag{89}$$

where t_0 is the gate working period, and the Max is taken with respect to all times and locations. Aharonov-Kitaev-Preskill (quant-ph/0510231) also showed that the local noise condition is satisfied if the system-bath coupling includes terms that are "always on," where each term acts on only a few system qubits. For either case, the local noise condition is proven by considering a formal expansion of U_{SB} in powers of H_{SB} and observing that at least one insertion of H_{SB} must occur at each faulty location.

The AGP argument proves that fault-tolerance works for fairly general non-Markovian noise models, but it has two big drawbacks. First, experimentalists do not measure Max $\|H_{SB,a}\|$, and it would be preferable to express the threshold condition in terms of a characterization of the noise that is more directly accessible in experiments. Second, there are reasonable noise models (like the spin-boson model) for which we expect fault-tolerant circuits to succeed, yet $\|H_{SB,a}\| = \infty$.

In the spin-boson model, $\langle H_{SB,a}\rangle$ can be large when the oscillator bath is highly excited, but when the bath temperature is small, the fluctuations of the oscillator's quadrature amplitude are limited; therefore we might expect to be able to use some variant of the AGP argument. One reason it is not straightforward to make this idea rigorous is that estimates of the "effective" value of $\|H_{SB,a}\|$ are sensitive to the very-high-frequency thermal fluctuations of the bath (Terhal and Burkard made this remark in quant-ph/0402104). We would like to incorporate into the analysis the idea that the high-frequency noise fluctuations tend to average out over the gate working period, assuming that the time dependence of H_S is sufficiently smooth.

If the pure state $|\psi\rangle$ is an eigenstate of H_B, then it is diagonal in the number-state basis, and therefore

$$\|H_{SB,a}|\psi\rangle\|^2 \le \frac{1}{4}\sum_k |g_k|^2 \langle\psi|a_k a_k^\dagger + a_k^\dagger a_k|\psi\rangle = \frac{1}{2}\sum_k |g_k|^2 \left\langle n_k + \frac{1}{2}\right\rangle \; . \tag{90}$$

Averaging over the thermal ensemble gives

$$\left\langle \|H_{SB,a}|\psi\rangle\|^2 \right\rangle_\beta \le \frac{1}{4}\int_0^\infty d\omega J(\omega)\coth(\beta\omega/2) = \frac{1}{8\pi}\int_0^\infty d\omega \left(\tilde{K}(\omega) + \tilde{K}(-\omega)\right) = \frac{1}{4}K(t=0) \; . \tag{91}$$

In the Ohmic case where $J(\omega) = A\omega e^{-\omega/\Lambda}$ and $\Lambda \gg \beta^{-1}$ is a high-frequency cutoff, the ω integral is dominated by frequencies near the cutoff:

$$\|H_{SB,a}\| \approx \sqrt{A}\ \Lambda\ . \tag{92}$$

By failing to take into account the softening of the high-frequency contribution due to coarse-graining in time, this estimate exaggerates the impact of the perturbation.

Suppose, then, that we consider a noise model in which the correlators of the bath are Gaussian. Should we be able to prove a threshold theorem under suitable conditions on the two-point function? In particular, we might expect to be able to find a simple characterization in the Ohmic case, or more generally if $\tilde{K}(\omega)$ approaches a finite nonzero limit as $\omega \to 0$. (Though $1/f$ noise may dominate at low frequency, we may assume that it has been suppressed through the appropriate use of spin-echo pulse sequences.)

We will need to determine how the bath correlators propagate as the level of a concatenated coding scheme changes. We don't expect to be able to do this exactly, but we can hope to establish inequalities that describe how the noise weakens with increasing level. Gaussianity will not be preserved under this coarse-graining step — that is, the correlations among faults in higher-level gadgets will not be completely characterized by the two-point correlation functions. Rather, connected correlations will be induced when we "integrate out" the fine-grained noise. However, these connected many-point functions will be systematically suppressed, so the effective noise model should remain well behaved (i.e., will not become too highly correlated) as the level of concatenation advances. (And since Gaussianity cannot be assumed at higher levels, we may as well relax that assumption at the lowest level – the argument, if it works at all, will apply as long as the bath's connected correlation functions are adequately suppressed.)

In particular, if the noise is correlated temporally but not spatially, we expect fault-tolerant protocols to be effective — if noise damages a qubit, things will not be much worse if further damage occurs to the same qubit within the same working period.

A reasonable (but too naive) guess for a threshold condition that should apply for Gaussian noise is

$$\eta \equiv \int_{\mathrm{Rec}} dt dx \int_{\mathrm{All}} dt' dx'\ K(t,x;t',x')\ \le\ \eta_{\mathrm{th}}\ . \tag{93}$$

Here we have indicated the dependence of the two-point correlation function on both time (t) and spatial position (x). To define the parameter η, the first argument of the correlator is integrated over a level-1 "rectangle," a gadget that realizes a fault-tolerant encoded gate, while the second argument is integrated over all locations in the entire circuit. This integral adds together contributions from all the possible "contractions" of locations in the rectangle with other locations, the contractions that can generate faults in the rectangle. Thus, if η is small, most level-1 rectangles have no faults, and only relatively few have two or more faults. Note that η is fairly insensitive to noise that has wavelength or period that is small compared to the spatial or temporal size of the rectangle.

A technical difficulty is that when we try to bound the trace norm of the sum of "bad" fault paths, it is not quite the quantity η that arises. Suppose that we consider the case of a Gaussian fluctuating magnetic field $f(t,x)$, and we express the output density operator produced by a noisy circuit as

$$\rho(t_f) = \int d\mu(f)\ \left(\sum \text{fault paths}\right)\rho(t_i)\left(\sum \text{fault paths}\right)^\dagger\ , \tag{94}$$

where $\int d\mu(f)$ denotes the normalized Gaussian integration over the field f. To upper bound the trace norm of a sum of fault paths that are bad at specified locations, we can pull the average over f outside the norm, and then for each fixed f bound the norm by following a variant of the AGP argument. But then in the upper bound the fluctuating magnetic field $f(t,x)$ becomes replaced by $|f(t,x)|$, and unfortunately the correlators of the absolute value $|f|$ are more ultraviolet sensitive than the correlators of f itself. For example, just as a Gaussian real variable x with mean zero satisfies $\langle\ |x|\ \rangle = \sqrt{2\langle x^2\rangle/\pi}$, so the one-point function of a Gaussian field is

$$\langle\ |f(t)|\ \rangle = \sqrt{\frac{2}{\pi}}\cdot\sqrt{K(0)}\ , \tag{95}$$

which blows up in the limit in which the noise is Markovian and $K(t)$ is a δ-function.

What is missing from the analysis and needs to be folded in somehow is the idea that because the system Hamiltonian is slowly varying we can filter out the high-frequency noise by smearing the insertion of $f(t)$ over the working period of a gate.

John Preskill, 3 July 2006
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