Solution to final exam problem I

(a) In electrostatics, the electric field is related to the electric potential according to \( \vec{E} = -\nabla V \). In spherical polar coordinates in the presence of rotational symmetry, this gives

\[
\vec{E} = -\frac{\partial V}{\partial r} \hat{r}.
\]  

(1)

Therefore the electric field for the given potential is

\[
\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{e^{-r/\lambda}}{r^2} \left(1 + \frac{r}{\lambda}\right) \hat{r}.
\]  

(2)

(b) By Gauss’s Law, the net charge contained within a sphere of radius \( R \) is the total electric flux through the surface of this sphere, times \( \epsilon_0 \). That is,

\[
Q_{\text{encl}} = \epsilon_0 \Phi = \epsilon_0 \int \vec{E} \cdot d\vec{A}.
\]  

(3)

The electric field in Eq. (2) is constant for constant \( R \) and always points in the radial direction. Therefore the electric flux is simply \( |\vec{E}(R)| \) times the surface area of a sphere of radius \( R \), \( 4\pi R^2 \). This gives

\[
Q_{\text{encl}} = Qe^{-R/\lambda} \left(1 + \frac{R}{\lambda}\right).
\]  

(4)

In the limit \( R \to 0 \), \( Q_{\text{encl}} \to Q \); whereas in the limit \( R \to \infty \), \( Q_{\text{encl}} \to 0 \).

(c) The electric field is defined so that the force on a particle of charge \( q \) in the presence of an electric field \( \vec{E} \) is \( \vec{F} = q\vec{E} \). Label the two \( Q \)-charges \( A \) and \( B \) and consider the force from \( A \) onto \( B \). Charge \( B \) attracts opposite charges to surround and screen it; however these charges are distributed symmetrically and do not exert a net force \( B \). Alternatively, Newton’s third law asserts that \( B \) cannot exert a force on itself. On the other hand, charge \( A \) exerts a force on \( B \), as do the charges that screen \( A \). These charges combine to create the electric field calculated in part (a). Therefore the force on \( B \) (located at \( r = d \)) is

\[
\vec{F} = \frac{Q^2}{4\pi\epsilon_0} \frac{e^{-d/\lambda}}{d^2} \left(1 + \frac{d}{\lambda}\right) \hat{r}.
\]  

(5)