The Operator Product Expansion for Deep Inelastic Scattering

Key variable $x = \frac{Q^2}{2p_\perp}$ is defined by

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Deep inelastic cross section is cross section in limit of $Q^2$ large and $x$ fixed. The nontrivial hadronic part of cross section came from

$$\sum_x (2\pi)^4 \delta^4 (q + p - px) \langle p | J^m(0) | x \rangle \langle x | J^v(0) | p \rangle$$

Spin averaging undisturbed here. Usually define hadronic tensor

$$W^{\mu \nu}(p, q) = \frac{i}{4\pi} \int d^4x e^{ix \cdot \phi} \langle p | [J^{\mu}(x), J^{\nu}(0)] | p \rangle$$

Inserting a complete set of states

$$W^{\mu \nu}(p, q) = i \sum_x (2\pi)^4 \delta^4 (q + p + px) \langle p | J^m(0) | x \rangle \langle x | J^v(0) | p \rangle$$

$$-\langle p | J^v(0) | x \rangle \langle x | J^m(0) | p \rangle$$

$$<x| J^m(0)| p > = <x| J^m(0)| p > e^{i(p-x)\cdot x}$$

$$e^{i(p-x)\cdot x}$$

$$W^{\mu \nu}(p, q) = \frac{i}{4\pi} \sum_x (2\pi)^4 \delta^4 (q + p - px) \langle p | J^m(0) | x \rangle \langle x | J^v(0) | p \rangle - e^{i(p-x)\cdot x}$$
\[ \langle p | J^\mu(0) | x \rangle \langle x | J^\nu(0) | p \rangle \]

The only states allowed here, \( p_x \neq 0 \), since baryon number is conserved. So for \( g_0 \geq 0 \) second delta function is not satisfied. For \( g_0 > 0 \) only first \( \delta (l) \) term contributes. Comparing with what we had before \( W = M W^\mu \), so we write

\[ W_{\mu \nu} = F_1 \left( -m_1 + \frac{g_0 g_2}{8} \right) + F_2 \left( \frac{F_{\mu \nu}}{P^2} \right) \]

The usual quantity you can derive Feynman rules for is the time ordered product matrix element

\[ T_{\mu \nu} = \langle p \bar{u} \gamma_{\mu} | T \{ J^\mu(0) J^\nu(0) \} | p \rangle \]

\[ = i \int d^4x \ e^{ig_2 \Phi(x)} \left[ T \{ J^\mu(0) J^\nu(0) \} \bar{u} \right] \]

\[ = i \int d^4x \ e^{ig_2 \Phi(x)} \left[ \Theta(2\pi) \langle p | J^\mu(0) J^\nu(0) | p \rangle \right. \]

\[ + \Theta(-2\pi) \langle p | J^\nu(0) J^\mu(0) | p \rangle \]
Feynman diagrams

Write

\[ T_{mn} = \left( -g_{\mu\nu} + \frac{q_{\mu} q_{\nu}}{q^2} \right) T_1 + \frac{1}{p \cdot q} \left( p_{\mu} - q_{\mu} \frac{p \cdot q}{q^2} \right) \left( p_{\nu} - \frac{q_{\nu}}{q^2} \right) T_2 \]

\[ T_{1,2} = T_{1,2} \left( q^2, p \cdot q \right) \]

Graph 1

\[ m_x^2 = (p + q)^2 = q^2 + 2p \cdot q + m_p^2 = -q^2 \left( \frac{1}{x} - 1 \right) + m_p^2 \]

Since \( m_x^2 \geq m_p^2 \)

Cut for \( x \in [0, 1] \), since \( q^2 < 0 \).

Graph 2

\[ m_x^2 = (p - q)^2 = -q^2 \left( \frac{1}{x} - 1 \right) + m_p^2 \geq m_p^2 \]

Cut for \( x \in [-1, 0] \)
Want to work on expression for time ordered product. Recall

$$\Theta(z) = \frac{-1}{2\pi i} \oint_{0} e^{-i\omega z}$$

For $z < 0$ close in upper half plane and get zero.
For $z > 0$ close in lower half plane and use residue theorem.

$$\Theta(z) = 0, \quad z < 0$$

$$\Theta(z) = -\frac{1}{2\pi i} (2\pi i) \frac{1}{z} = 1, \quad z > 0$$

Let's use that expression for $T_{\nu}$

$$T_{\nu} = \frac{-1}{2\pi i} \int_{0} d\omega \int d^4 z e^{i\omega z} \frac{\langle p | J_{\nu}(z) J_{\nu}(0) | p \rangle}{\omega + i\epsilon}$$

Next insert a complete set of states between the currents.
\[
T_n = \frac{1}{2\pi} \sum \int \int \int \int e^{i \varphi_2 - i \omega \varphi_2} \left< \rho \left| J_n(\varphi) \right| x(p_\varphi) \right> \frac{\left< x(p_\varphi) \left| J_n(\varphi) \right| x(p_\varphi) \right>}{\omega + i \epsilon}
\]

\[
\left< x(p_\varphi) \left| J_n(\varphi) \right| x(p_\varphi) \right>
\]

\[
= \frac{1}{2\pi} \sum \int \int \int \int e^{i \varphi_2 - i \omega \varphi_2} e^{i \hat{P}_2} \left< \rho \left| e^{i J_n(\varphi) e^{-i \hat{P}_2}} \right| \rho \right> \frac{\left< x(p_\varphi) \left| e^{i J_n(\varphi) e^{-i \hat{P}_2}} \right| x(p_\varphi) \right>}{\omega + i \epsilon}
\]

\[
\left< x(p_\varphi) \left| e^{i \hat{P}_2} J_n(\varphi) e^{-i \hat{P}_2} \right| x(p_\varphi) \right>
\]

\[
= \frac{1}{2\pi} \sum \int \int \int \int e^{i \varphi_2 - i \omega \varphi_2} \left< \rho \left| e^{i \hat{P}_2} J_n(\varphi) e^{-i \hat{P}_2} \right| x(p_\varphi) \right> \frac{\left< x(p_\varphi) \left| e^{i \hat{P}_2} J_n(\varphi) e^{-i \hat{P}_2} \right| x(p_\varphi) \right>}{\omega + i \epsilon}
\]

\[
\left< x(p_\varphi) \left| e^{i \hat{P}_2} J_n(\varphi) e^{-i \hat{P}_2} \right| x(p_\varphi) \right>
\]
\[ \text{Do } d \frac{\partial}{\partial z} \text{ integrals} \]
\[ = -i \sum_{x} \frac{(2 \pi)^{3} \delta^{3}(z - px + p)}{8^{0} + p^{0} + px + i \epsilon} \langle \rho | J_{\mu}(\omega) | x(\omega) \rangle \langle x(\omega) | T_{\mu}(0) | \rho \rangle \]
\[ - \sum_{x} \frac{(2 \pi)^{3} \delta^{3}(z - px + p)}{8^{0} + p^{0} + px + i \epsilon} \langle \rho | J_{\mu}(\omega) | x(\omega) \rangle \langle x(\omega) | T_{\mu}(0) | \rho \rangle \]

Now
\[ \frac{1}{i \omega + i \epsilon} = \text{PP} \left( \frac{1}{\omega} \right) - i \pi \delta(\omega) \]

Some lan

\[ T_{\mu} = \text{principal part piece} \]
\[ + i \pi \sum_{x} (2 \pi)^{3} \delta^{3}(z - px + p) \langle \rho | J_{\mu}(\omega) | x(\omega) \rangle \langle x(\omega) | T_{\mu}(0) | \rho \rangle \]
\[ + i \pi \sum_{x} (2 \pi)^{3} \delta^{3}(z - px + p) \langle \rho | J_{\mu}(\omega) | x(\omega) \rangle \langle x(\omega) | T_{\mu}(0) | \rho \rangle \]

\[ q^{0} > 0 \quad \text{second term does not contribute} \]

\[ \text{Im } T_{\nu} = 2 \pi W_{\nu} \]
\[ \text{Im } T_{\mu} = 2 \pi F_{\mu} \]
Consider the actual product of local operators
squared by \( z \) :

\[
T[O_\alpha(2) O_\beta(0)]
\]

For small \( z \) the product is practically a sum over a product of operators, can be written

\[
T[O_\alpha(2) O_\beta(0)] = \sum_{\alpha} \text{Caut} O_\alpha(0)
\]

Coefficients \( \text{Caut} \) depend on \( z \). Low momentum (compared with \( z \)) matrix elements of \( \text{LHS} \) are approximated to those of \( \text{RHS} \). The coefficients \( \text{Caut} \) don't depend on all matrix elements depending on \( \text{Caut} \) in operator equal contractions. In \( \text{QCD} \) the coupling constant is small because of asymptotic freedom. Can compute coefficients functions of small \( z \) momentum perturbation theory with great gluon states. The momentum space version of \( \text{QCD} \)

\[
\int dz e^{i z - \lambda} T[O_\alpha(2) O_\beta(0)]
\]

\[
= \sum_\alpha \text{Caut}(z) O_\alpha(0)
\]

valid if large \( \lambda \). Valid for all matrix elements provided \( z \) is much than external momenta.

We will use the momentum space form of
Now let's come on case. The operators might be non-trivial and will have a spin. Convoluted spin $n$ + dimension can be deduced on some matter & nucleon like combination $m_{n-1}$. 

$$\langle 0 | \mathcal{O}_{m...n} | \rho \rangle \propto \frac{d_{m-2}}{\sqrt{m_{n...m}}}$$

$$\langle \rho | \rho \rangle = (\alpha \pi)^3 L^3 (\pi)$$

$$\langle \rho | | = \frac{\pi}{2} \pi$$

$$\langle \mathcal{T}_{m...n} | \mathcal{O}_{n,n} | \rho \rangle \propto \left( \frac{\alpha \pi}{\pi} \right)^{2-d} \frac{d_{m-2}}{\frac{d_{m}}{m}}$$

$$\mathcal{T}_{m...n} \propto \frac{d_{n-1}}{m} \pi$$

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$$+ \alpha \pi = \text{dimension} \times \text{spin}$$

Lowest twist dominates.

Want to make gauge invariant operators.
Conventional bag 1 twist 2 quark operators that contribute in deep inelastic electron processes (spin averaged): \[
O_{q,v} = \frac{1}{2} \left( \frac{i}{2} \right)^{n-2} \sum_{\text{h.m.}} \langle \bar{Q} \gamma^{\mu_1} D_{\mu_2} \cdots D_{\mu_{n-2}} G \gamma^{\nu} \rangle
\]
\[
\rightarrow D^x B = \tilde{A} D^x B - \tilde{A} D^x B \quad \text{no chiral limit change}
\]

Also gluon operators:
\[
O_{g,v} = -\frac{1}{2} \left( \frac{i}{2} \right)^{n-2} \sum_{\text{h.m.}} \langle G_{\mu_1} D_{\mu_2} \cdots D_{\mu_{n-1}} G_{\nu} \rangle
\]

We will work to lowest order in $\alpha_s$ and the gluon operators don't arise. Most general form is then consistent with current conservation.

\[
\tau_{q,v} = \sum_{m,n} \left( -\delta_{m1} + \epsilon_{m2} \epsilon_{n2} \right) \frac{2 \delta_{m1} \delta_{n1}}{\Delta_{q,v}} \frac{1}{(-\Delta_{q,v})^n}
\]
\[
\Theta_{q,v} + \sum_{m,n} \left( \delta_{m1} - \epsilon_{m2} \epsilon_{n2} \right) \frac{2 \delta_{m1} \delta_{n1}}{\Delta_{q,v}} \frac{1}{(-\Delta_{q,v})^n}
\]

Determine $C_{q,v}^{\mu_1}$, $C_{g,v}^{\mu_1}$ from matching and check using quark matrix elements.