1. State and prove Wick’s Theorem for anticommuting Fermi fields.

2. Show that the disconnected vacuum diagrams cancel out from the diagrammatic expansion of the two-point function

$$\langle \Omega | T\{\phi(x)\phi(y)\} | \Omega \rangle$$

in $\lambda \phi^4$ theory, up to and including terms of order $\lambda^2$.

3. a) Show that Wick’s Theorem can be written as

$$T[\phi_1(x_1)\ldots\phi_1(x_n)] = \exp \left[ \frac{1}{2} \int d^4x d^4y D_F(x-y) \frac{\delta}{\delta \phi_1(x)} \frac{\delta}{\delta \phi_1(y)} \right] \phi_1(x_1)\ldots\phi_1(x_n),$$

where the functional derivative $\frac{\delta}{\delta \phi_1(x)}$ satisfies

$$\frac{\delta \phi_1(x)}{\delta \phi_1(y)} = \delta^4(x-y).$$

b) Consider the following “interacting” field theory:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \rho(x) \phi(x)$$

where $\rho(x)$ is a c-number function with $\rho(x) \to 0$ as $|x| \to \infty$. Using the previous result, show that.

$$U(\infty, -\infty) = T\exp \left[ -i \int_{-\infty}^{\infty} dt H_I(t) \right]$$

is given by

$$U(\infty, -\infty) = \exp \left[ -\frac{1}{2} \int d^4x d^4y \rho(x)\rho(y) D_F(x-y) \right] \exp \left[ -i \int d^4z \rho(z) \phi_I(z) \right].$$

c) Compute

$$P(n) = \frac{1}{n!} \int \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \ldots \frac{d^3k_n}{(2\pi)^3 2E_{k_n}} | \langle k_1\ldots k_n | U(\infty, -\infty) | 0 \rangle |^2.$$ 

This is the probability of finding $n$ particles in the far future starting with the vacuum in the far past. (Since $\rho(x) \to 0$ as $t \to \pm \infty$, the initial/final states coincide with those of the free field theory. Show that $n$ is Poisson distributed:

$$P(n) = \frac{\lambda^n}{n!} e^{-\lambda},$$

with

$$\lambda = \int \frac{d^3k}{(2\pi)^3 2E_k} | \tilde{\rho}(\vec{k}, E_k) |^2.$$ 

Here

$$\tilde{\rho}(\vec{k}, E_k) = \int d^4x \rho(x) e^{ik \cdot x},$$

with $k^0 = E_k$. 

1