Phys 125a: Homework 6. Due Nov. 30, 2005

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Problem 1

(a) Derive the recursion relations

$$P'_{n-1}(x) = -nP_n(x) + xP'_n(x)$$
$$(1 - x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x)$$
$$(1 - x^2)P'_n(x) = (n + 1)xP_n(x) - (n + 1)P_n(x)$$

(b) Show that for $m \geq 0$

$$\int_{-1}^{1} dx P_\ell^m(x) P_{\ell'}^m(x) = \frac{2}{2\ell + 1} \frac{(\ell + m)!}{(\ell - m)!} \delta_{\ell\ell'}$$

Problem 2 (Exercises 12.5.4) (1) Argue that the eigenvalues of $J_x^{(j)}$ and $J_y^{(j)}$ are the same as those of $J_z^{(j)}$, namely, $j\hbar, (j-1)\hbar, \ldots, (-j\hbar)$. Generalize the result to $\hat{\theta} \cdot \mathbf{J}^{(j)}$.

(a) Show that

$$(J - j\hbar)[J - (j-1)\hbar][J - (j-2)\hbar] \ldots (J + j\hbar) = 0$$

where $J \equiv \hat{\theta} \cdot \mathbf{J}^{(j)}$. (Hint: In the case $J = J_z$ what happens when both sides are applied to an arbitrary eigenket $|jm\rangle$? What about an arbitrary superpositions of such kets?)

(b) It follows from (2) that J^{2j+1} is a linear combination of $J^0, J^1, \ldots, J^{2j}$. Argue that the same goes for $J^{2j+k}, k = 1, 2, \ldots$.

Problem 3 (*Exercise 12.6.1.*) A particle is described by the wave function

$$\Psi_E(r, \theta, \phi) = Ae^{-r/a_0} \quad (a_0 = \text{const})$$

(a) What is the angular momentum content of the state?

(b) Assuming Ψ_E is an eigenstate in a potential that vanishes as $r \to \infty$, find E. Match leading terms in Schrödinger's equation.)

(c) Having found E, consider finite r and find $V(r)$.

Problem 4 (*Exercise 12.5.13.*) Consider a particle in a state described by

$$\Psi = N(x + y + 2z)e^{-ar}$$

where N is a normalization factor.

(a) Show, by rewriting the $Y_1^{\pm 1, 0}$ functions in terms of x, y, z, and r, that

$$Y_1^{\pm 1} = \mp \left(\frac{3}{4\pi}\right)^{1/2} \frac{x \pm iy}{2^{1/2} r}$$
$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \frac{z}{r}.$$

(b) Using this result, show that for a particle described by ψ above, $P(l_z = 0) = 2/3; P(l_z = +\hbar) = 1/6 = P(l_z = -\hbar)$.