Isotropic Oscillator

The isotropic Harmonic oscillator is described by Hamiltonian

$$H = \frac{P_x^2 + P_y^2 + P_z^2}{2m} + \frac{1}{2} m\omega^2 (X^2 + Y^2 + Z^2)$$

$$\psi_{Elm} = \frac{U_{El}(r)}{r} Y_l^m(\theta, \phi)$$

$$\left\{ \frac{d^2}{dr^2} + \frac{2m}{\hbar^2} \left[E - \frac{1}{2} m\omega^2 r^2 - \frac{l(l+1)\hbar^2}{2mr^2} \right] \right\} U_{El} = 0$$

$$E = \lambda \hbar\omega \qquad y = \left(\frac{m\omega}{\hbar}\right)^{1/2} r$$

$$\Rightarrow \frac{d}{dr} = \left(\frac{m\omega}{\hbar}\right)^{1/2} \frac{d}{dy}$$

$$\left\{ \left(\frac{m\omega}{\hbar}\right) \frac{d^2}{dy^2} + \frac{m\omega}{\hbar}(2\lambda) - \left(\frac{m^2\omega^2}{\hbar^2}\right)\left(\frac{\hbar}{m\omega}\right) y^2 - \frac{l(l+1)}{y^2}\left(\frac{m\omega}{\hbar}\right) \right\} U_{El}(y) = 0$$

$$\left\{ \frac{d^2}{dy^2} + 2\lambda - y^2 - \frac{l(l+1)}{y^2} \right\} U_{El}(y) = 0$$

as $y \to \infty$ $U_{El}(y) \sim e^{-y^2/2}$ $\xrightarrow{\text{so put}}$ + sign also possible but not normalizable

$$U(y) = e^{-y^2/2} v(y)$$

$$\frac{d}{dy} U(y) = -y e^{-y^2/2} v(y) + e^{-y^2/2} v'(y)$$

$$\frac{d^2}{dy^2} U(y) = -e^{-y^2/2} v(y) - 2y e^{-y^2/2} v'(y) + e^{-y^2/2} v''(y) + y^2 e^{-y^2/2} v(y)$$

$$v'' - 2yv' + (2\lambda - 1)v - \frac{\ell(\ell+1)}{y^2}\, v = 0$$

Seek series soln

$$v(y) = y^{\ell+1}\sum_{j=0}^{\infty} c_j y^j = \sum_{j=0}^{\infty} c_j y^{j+\ell+1}$$

from previous work

$$\sum_{j=0}^{\infty}\Big[c_j (j+\ell+1)(j+\ell) y^{j+\ell-1} + c_j(-2)(j+\ell+1) y^{j+\ell+1} + (2\lambda-1) c_j y^{j+\ell+1} - \ell(\ell+1) c_j y^{j+\ell-1} \Big]$$

Now coeff: $\quad y^{\ell-1}\left[c_0\big((\ell+1)\ell - \ell(\ell+1)\big)\right] = 0$

Coeff $\quad y^{\ell}\left[c_1\big((\ell+2)(\ell+1) - \ell(\ell+1)\big)\right] = 0 \Rightarrow c_1 = 0$

Relabel series

$$\sum_{j=0}^{\infty} y^{j+\ell+1}\Big(c_{j+2}\big((j+\ell+3)(j+\ell+2) - \ell(\ell+1)\big) + c_j\big(-2(j+\ell+1) + 2\lambda - 1\big)\Big) = 0$$

$$c_{j+2} = 2c_j\left[\frac{j+\ell+3/2-\lambda}{(j+\ell+3)(j+\ell+2) - \ell(\ell+1)}\right]$$

Since $c_1 = 0$ $\quad j = 2k$ & series terminates if

$$\lambda = 2k + \ell + 3/2 \qquad k = 0, 1, 2, \ldots$$

Where principal quantum number $n = 2k + \ell$

$$\lambda = n + 3/2 \qquad \ell = n - 2k = n, n-2, \ldots 1 \text{ or } 0$$

Here are a few eigenstates

$n=0 \quad l=0 \quad m=0$

$n=1 \quad l=1 \quad m=1,0,-1$

$n=2 \quad l=0,2 \quad m=0, \ \pm 2, \pm 1, 0$

$n=3 \quad l=1,3 \quad m=\pm 1, 0; \ \pm 3, \pm 2, \pm 1, 0$

etc.

The Hydrogen Atom

Two body with electron of mass m proton of mass M. Electron of charge $-e$ & proton of charge $+e$. Go to CM frame. Reduced mass

$$\mu = \frac{Mm}{M+m} \simeq m \qquad \frac{m}{M} \simeq \frac{1}{2{,}000}$$

$$V = -\frac{e^2}{r}$$

So the Schrodinger equation is

$$\left\{ \frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}\left[E + \frac{e^2}{r} - \frac{l(l+1)\hbar^2}{2\mu r^2}\right]\right\} u_{El} = 0$$

$$\psi_{Elm} = R_{El}(r)\, Y_l^m(\theta,\phi) = \frac{u_{El}(r)}{r}\, Y_l^m(\theta,\phi)$$

Now we are interested in bound states with $E<0$. So lets introduce variable

$$\rho = \left(\frac{-2\mu E}{\hbar^2}\right)^{1/2} r$$

$$\left\{\left(\frac{-2mE}{\hbar^2}\right)\frac{d^2}{d\rho^2} + \frac{2mE}{\hbar^2} + \left(\frac{2mE}{\hbar^2}\right)\frac{e^2}{E\rho}\left(\frac{-2mE}{\hbar^2}\right)^{1/2}\right.$$

$$\left. - \frac{l(l+1)}{\rho^2}\left(\frac{-2mE}{\hbar^2}\right)\right\} U_{E,l}(\rho) = 0 \qquad \lambda = \left(\frac{2m}{-E\hbar^2}\right)$$

$$\left(\frac{d^2}{d\rho^2} - 1 + \frac{e^2\lambda}{\rho} - \frac{l(l+1)}{\rho^2}\right) U_{El}(\rho) = 0$$

Next lets look that at ∞ $U_{El}(\rho) \sim e^{-\rho}$ (or e^{ρ})
so we

$$U_{El}(\rho) = v_{El}(\rho) e^{-\rho}$$

$$\frac{dU}{d\rho} = v' e^{-\rho} - v e^{-\rho}$$

$$\frac{d^2U}{d\rho^2} = v'' e^{-\rho} - 2v' e^{-\rho} + v e^{-\rho}$$

$$\frac{d^2v}{d\rho^2} - 2\frac{dv}{d\rho} + \left[\frac{e^2\lambda}{\rho} - \frac{l(l+1)}{\rho^2}\right] v = 0$$

Take

$$v_{El} = \rho^{l+1}\sum_{k=0}^{\infty} C_k \rho^k \quad \text{+ plug into above}$$

$$\sum_{k=0}^{\infty}\Big[C_k(k+l+1)(k+l)\rho^{k+l-1} - 2C_k(k+l+1)\rho^{k+l}$$
$$+ e^2\lambda C_k \rho^{k+l} - l(l+1)C_k\rho^{k+l-1}\Big]$$

$$\sum_{k=0}^{\infty}\Big\{C_k\underbrace{\left[(k+l+1)(k+l) - l(l+1)\right]}_{=0}\rho^{k+l-1} + \left[e^2\lambda - 2(k+l+1)\right]C_k\rho^{k+l}$$

$$\sum_{k=0}^{\infty} \Big\{ C_{k+1} \left[(k+l+2)(k+l+1) - l(l+1) \right] + \left[e^2\lambda - 2(k+l+1) \right] C_k \Big\} \rho^{k+l} = 0$$

$$\frac{C_{k+1}}{C_k} = \left[\frac{-e^2\lambda + 2(k+l+1)}{(k+l+2)(k+l+1) - l(l+1)} \right]$$

Again for suitable behavior at ∞ need series to terminate

$$+e^2\lambda = 2(k+l+1)$$

$$e^2\left(\frac{2m}{-E\hbar^2}\right)^{1/2} = 2(k+l+1)$$

$$E = -\frac{me^4}{2\hbar^2(k+l+1)^2} \qquad k = 0, 1, 2, \ldots \quad l = 0, 1, 2 \ldots$$

Now introduce principal quantum no. $n = k+l+1$

$$E_n = \frac{-me^4}{2\hbar^2 n^2} \qquad n = 1, 2, 3.$$

$$l = n-k-1 = n-1, n-2, \ldots 0$$

States of different l are degenerate. Degeneracy at each n

$$\sum_{l=0}^{n-1} \underbrace{(2l+1)}_{m \text{ values}} = n^2$$

Rydberg $R_y = \dfrac{me^4}{2\hbar^2} \rightarrow$ natural unit of energy