The order of operators typically matter

$$\Omega\Lambda - \Lambda\Omega \equiv [\Omega, \Lambda]$$

is called the commutator of Ω and Λ. Typically it doesn't vanish.

Useful commutator relations

$$[\Omega, \Lambda\theta] = \Lambda[\Omega, \theta] + [\Omega, \Lambda]\theta$$

$$[\Lambda\Omega, \theta] \quad \Lambda[\Omega, \theta] + [\Lambda, \theta]\Omega$$

check for $\Lambda[\Omega, \theta] + [\Omega, \Lambda]\theta$
$= \cancel{\Lambda\Omega\theta} - \Lambda\theta\Omega + \Omega\Lambda\theta - \cancel{\Lambda\Omega\theta}$
$= [\Omega, \Lambda\theta]$ ✓

The inverse of a linear operator Ω is denoted by Ω^{-1} and it satisfies

$$\Omega^{-1}\Omega = \Omega\Omega^{-1} = 1$$

If $\det \Omega \neq 0$ then operator has inverse.
not every operator has an inverse but if $\Omega|V\rangle = |0\rangle$ implies $|V\rangle = |0\rangle$ then the operator Ω has an inverse.

For a product of operators $(\Lambda\Omega)^{-1} = \Omega^{-1}\Lambda^{-1}$

since

$$(\Lambda\Omega)^{-1}(\Lambda\Omega) = \Omega^{-1}\underbrace{\cancel{\Lambda^{-1}\Lambda}}_{1}\Omega = \Omega^{-1}\Omega = 1$$

Suppose we have an o.n. basis

$$|V\rangle = \sum_i v_i |i\rangle$$

$$\Omega|V\rangle = \sum_i v_i' |i\rangle$$

But $\Omega|V\rangle = \sum_i v_i \, \Omega|i\rangle$

$$= \sum_{i,j} v_i |j\rangle\langle j|\Omega|i\rangle$$

$$\sum_i v_i' |i\rangle = \sum_{j,i} v_j |i\rangle\langle i|\Omega|j\rangle$$

$$v_i' = \sum_j \langle i|\Omega|j\rangle v_j = \sum_j \Omega_{ij} v_j$$

matrix element of operator

$$\begin{bmatrix} v_1' \\ \vdots \\ v_n' \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \cdots & \Omega_{1n} \\ \vdots & & \\ \Omega_{n1} & \cdots & \Omega_{nn} \end{bmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

matrix representation of operator

Note matrix elements of identity operator are

$$\langle i|I|j\rangle = \delta_{ij} \qquad \text{Kronecker delta}$$

Next consider arbitrary on basis

$$|V\rangle = \sum_i |i\rangle\langle i|V\rangle$$

$$= \sum_i \left(|i\rangle\langle i|\right) |V\rangle$$

So we can view $\sum_i |i\rangle\langle i|$ as a linear operator identity

$$I = \sum_i |i\rangle\langle i| = \sum_i \mathbb{P}_i$$

$$\mathbb{P}_i = |i\rangle\langle i|$$

Note $\mathbb{P}_i \mathbb{P}_j = \delta_{ij} \mathbb{P}_j$

$$\mathbb{P}_i |V\rangle = |i\rangle v_i$$

$$\langle V| \mathbb{P}_i = v_i^* \langle i|$$

Now consider the product of two linear operators $\Omega \Lambda$. $(\Omega\Lambda)|V\rangle = \Omega(\Lambda|V\rangle)$.

$$(\Omega\Lambda)_{ij} = \langle i|\Omega\Lambda|j\rangle$$

$$= \langle i|\Omega I \Lambda|j\rangle$$

$$= \sum_k \langle i|\Omega|k\rangle\langle k|\Lambda|j\rangle$$

$$= \sum_k \Omega_{ik} \Lambda_{kj}$$

matrix multiplication

We have that linear operators act on kets you can also view them as acting on bra's

$$(\langle V_i|\alpha + \langle V_j|\beta)\Omega$$

$$= \alpha\langle V_i|\Omega + \beta\langle V_j|\Omega\rangle$$

Now sometimes we write $\Omega|V\rangle = |\Omega V\rangle$. The corresponding bra vector is $\langle \Omega V|$ which we can write as some linear operator acting on $\langle V|$. That linear operator is the adjoint of Ω

$$\langle \Omega V| = \langle V|\Omega^{\dagger}$$

What are its matrix elements

$$\langle i|\Omega^{\dagger}|j\rangle = \langle \Omega i|j\rangle = \langle j|\Omega i\rangle^{*}$$
$$= \langle j|\Omega|i\rangle^{*} = \Omega^{*}_{ji}$$

$$(\Omega^{\dagger})_{ij} = \Omega^{*}_{ji} \quad (\text{Note } (\Omega^{\dagger})^{\dagger} = \Omega)$$

The adjoint of a product is the product of the adjoints in reverse

$$(\Omega\Lambda)^{\dagger} = \Lambda^{\dagger}\Omega^{\dagger}$$

Obvious from matrix multiplication but also

$$\langle \Omega\Lambda V| = \langle \Omega(\Lambda V)| = \langle \Lambda V|\Omega^{\dagger} = \langle V|\Lambda^{\dagger}\Omega^{\dagger}.$$

<u>Hermitian and Unitary operators</u>

A linear operator Ω is *Hermitian* if $\Omega^{\dagger} = \Omega$ and anti Hermitian if $\Omega^{\dagger} = -\Omega$. We can decompose any operator into its Hermitian + anti Hermitian parts

$$\Omega = \left[\frac{\Omega + \Omega^{\dagger}}{2}\right] + \left[\frac{\Omega - \Omega^{\dagger}}{2}\right]$$

like decomposing a complex number into its real + imaginary parts. A unitary operator U satisfies

$$UU^\dagger = U^\dagger U = I$$

This is analogous to complex number of unit modulus which satisfy $zz^* = 1$. The special idea about unitary operators is that when they act on vectors they preserve the inner product between the

$$|V_1'\rangle = U|V_1\rangle \quad , \quad |V_2'\rangle = U|V_2\rangle$$

$$\langle V_1'|V_2'\rangle = \langle UV_1 | UV_2\rangle = \langle V_1|U^\dagger U|V_2\rangle$$

$$= \langle V_1|V_2\rangle$$

If one looks at $n \times n$ matrix of unitary operator the columns can be thought of as vectors. These columns are O.N. (so are the rows).

$$\mathbb{1} = U^\dagger U$$

$$\langle i|\mathbb{1}|j\rangle = \sum_k \langle i|U^\dagger|k\rangle\langle k|U|j\rangle$$

$$\delta_{ij} = \sum_k U_{ki}^* U_{kj}$$

Suppose we subject all vectors in our space to a unitary transformation $|V\rangle \to U|V\rangle$. Under this transformation the matrix elements of any linear operator become

$$\langle V'|\Omega|V\rangle \to \langle UV'|\Omega|UV\rangle$$

$$= \langle V'|U^\dagger \Omega U|V\rangle$$

So the same change can be achieved by leaving the vectors alone and transforming the operator

$$\Omega \to U^\dagger \Omega U$$

First are called an active transformation + second passive

Consider some linear operator Ω acting on an arbitrary non zero ket $|V\rangle$

$$\Omega |V\rangle = |V'\rangle$$

Usually $|V'\rangle$ is very different from $|V\rangle$. But every operator has certain vectors called eigenvectors for which action is a simple rescaling,

$$\Omega |V\rangle = \omega |V\rangle$$

We call $|V\rangle$ an eigenvector of Ω and ω its eigenvalue. I want to remind you of the linear algebra that lets you take an operator and find its eigenvectors + eigenvalues. Well in some cases its totally trivial. For identity

$$I |V\rangle = |V\rangle$$

for all $|V\rangle$. For a projection $P_i = |i\rangle\langle i|$ ($|i\rangle$ are normalized but

$$P_i |i\rangle = 1|i\rangle$$
$$P_i |j\rangle = 0|j\rangle \quad \text{for any } |j\rangle \text{ orthogonal to } |i\rangle$$

How does it work more generally. We can view $\Omega|V\rangle = \omega|V\rangle$ as

$$(\Omega - \omega I)|V\rangle = |0\rangle$$

So $\Omega - \omega I$ has a non zero vector that is mapped to zero. That means $\Omega - \omega I$ is not invertible and its determinant vanishes

$$\det(\Omega - \omega I) = 0 \quad \left(\begin{array}{l}\text{polynomial of degree} \\ n \text{ if view } \Omega \text{ as } n\times n \\ \text{matrix}\end{array}\right)$$

This equation determines the eigenvalues ω. We still need to find the eigenvector

$$(\Omega - \omega \mathbb{1})|V\rangle = 0$$

$$\Rightarrow \sum_j \langle i|\Omega - \mathbb{1}\omega)|j\rangle\langle j|V\rangle = 0$$

$$\Rightarrow \sum_j (\Omega_{ij} - \omega\delta_{ij}) v_j = 0$$

example:

$$\Omega = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

This is actually a unitary operator $\Omega\Omega^\dagger = \mathbb{1}$.

$$\det(\Omega - \omega \mathbb{1}) = \begin{vmatrix} 1-\omega & 0 & 0 \\ 0 & -\omega & -1 \\ 0 & 1 & -\omega \end{vmatrix}$$

$$= (1-\omega)\begin{vmatrix} -\omega & -1 \\ 1 & -\omega \end{vmatrix} = (1-\omega)(\omega^2+1)$$

Three eigenvalues $\omega = 1$, $\omega = i$, $\omega = -i$. Lets find eigenvectors

(i) $\omega = 1$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} v_2 + v_3 = 0 \\ v_2 - v_3 = 0 \end{matrix} \Rightarrow v_2, v_3 = 0 \Rightarrow |\omega = 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(ii) $\omega = i$

$$\begin{bmatrix} 1-i & 0 & 0 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$v_1 = 0$, $\left.\begin{matrix} -iv_2 - v_3 = 0 \\ v_2 - iv_3 = 0 \end{matrix}\right\}$ same equation

$$v_2 = i v_3$$

$$|\omega = i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$$

(iii) $\omega = -i$

Show $|\omega = -i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$

An important property of the eigenvalues of a Hermitian matrix is that they are real

$$\Omega|\omega\rangle = \omega|\omega\rangle$$

$$\langle\omega|\Omega|\omega\rangle = \omega\langle\omega|\omega\rangle$$

$$\langle\omega|\Omega^{\dagger}|\omega\rangle = \langle\omega|\Omega|\omega\rangle^{*}$$

$$= \omega^{*}\langle\omega|\omega\rangle$$

$\Omega = \Omega^{\dagger}$ subtract $(\omega-\omega^{*})\langle\omega|\omega\rangle = 0 \Rightarrow \omega = \omega^{*}$

Suppose two eigenvalues of a Hermitian matrix are different

$$\Omega|\omega_i\rangle = \omega_i|\omega_i\rangle$$
$$\Omega|\omega_j\rangle = \omega_j|\omega_j\rangle$$

$$\langle\omega_j|\Omega|\omega_i\rangle = \omega_i\langle\omega_j|\omega_i\rangle$$

$\Omega^{\dagger} = \Omega$ $\quad = \omega_j\langle\omega_j|\omega_i\rangle$

$\Rightarrow \omega_i \neq \omega_j$ then $\langle\omega_j|\omega_i\rangle = 0$

So eigenvectors corresponding to different eigenvalues are orthogonal. For there may be several, say, m_i eigenvectors for the single eigenvalue ω_m. Call them $|\omega_m;1\rangle, \ldots |\omega_m;m\rangle$. Note, [illegible] these are not necessarily orthogonal. They are a basis for a subspace of it. Perform Gram Schmidt on this basis. So they can be chosen orthogonal. Hence every hermitian matrix has an o.n basis of eigenvectors

Well we know a lot about eigenvalues & eigenvectors for Hermitian operators. What about unitary operators. Suppose

$$U|U_i\rangle = U_i|U_i\rangle$$

$$U|U_j\rangle = U_j|U_j\rangle$$

Now $\langle U_j|U^\dagger U|U_i\rangle = U_i U_j^* \langle U_j|U_i\rangle$

$\Rightarrow \langle U_j|U_i\rangle(1 - U_i U_j^*) = 0$

For $i=j$ get $|U_i|^2 = 1$ and for different eigenvalues $U_i \neq U_j$, $\langle U_j|U_i\rangle = 0$. Also a lot like the Hermitian case.

Consider a Hermitian operator Ω on $V^n(C)$ represented by a matrix in some o.n. basis $|1\rangle, \ldots |i\rangle, \ldots |n\rangle$. In o.n. basis $|\omega_1\rangle, \ldots |\omega_i\rangle, \ldots |\omega_n\rangle$ it is diagonal. There is some operator that induces this change of basis

$$|\omega_i\rangle = U|i\rangle$$

Check its unitary and

$$U^\dagger \Omega U$$

$\langle i|U^\dagger U|j\rangle$
$= \langle\omega_i|\omega_j\rangle$
$= \delta_{ij}$
$U^\dagger U = 1$

is a diagonal matrix so

$$\langle j|U^\dagger \Omega U|i\rangle = \langle\omega_j|\Omega|\omega_i\rangle = \delta_{ij}$$

Every Hermitian matrix can be diagonalized by a unitary matrix