$$(1-2xt+t^2)^{-1} = \left[\sum_{n=0}^{\infty} P_n(x)\, t^n\right]^2$$

$$\int_{-1}^{1} dx \,\frac{1}{(1-2xt+t^2)^{-1}} = \sum_{n=0}^{\infty} \int_{-1}^{1} dx\, P_n^2(x)\, t^{2n}$$

$$\Rightarrow \frac{-1}{2t} \ln\left[\frac{1-2t+t^2}{1+2t+t^2}\right] = \sum_{n=0}^{\infty} \int_{-1}^{1} dx\, P_n^2(x)\, t^{2n}$$

$$\Rightarrow -\frac{1}{t}\left(\ln[1-t] - \ln[1+t]\right)$$

doubles odd terms

$$= \frac{1}{t} \sum_{n=1}^{\infty} \left(\frac{t^n}{n} + (-1)^{n+1}\frac{t^n}{n}\right)$$

$$= 2 \sum_{n=0}^{\infty} \left(\frac{t^{2n}}{2n+1}\right)$$

$$\int_{-1}^{1} dx\, P_n^2(x) = \frac{2}{2n+1}$$

One more relation I would like to show. First I need a result about double sums

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{n,k} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} a_{n-k,k}$$

LHS

$$\underset{(n=0)}{a_{0,0}} + \underset{(n=1)}{(a_{1,0}+a_{11})} + \underset{(n=2)}{(a_{2,0}+a_{21}+a_{22})} + \underset{(n=3)}{(a_{30}+a_{31}+a_{32}+a_{33})} + \dots$$

RHS

$$\underset{(n=0)}{a_{0,0}} + \underset{n=1}{(a_{1,0})} + \underset{n=2}{(a_{2,0}+a_{1,1})} + \underset{n=3}{(a_{3,0}+a_{2,1})} + \underset{n=4}{(a_{4,0}+a_{3,1}+a_{2,2})} + \dots$$

Next

$$(1-2xt+t^2)^{-1/2}$$

$$= 1 + \frac{1}{2}(2xt - t^2) + \frac{\frac{1}{2}(\frac{1}{2}+1)}{2!}(2xt-t^2)^2 + \frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)}{3!}(2x-t)^3$$

$$\frac{\frac{1}{2}(\frac{1}{2}+1)}{2!} = \frac{1\cdot 3}{2!2^2} = \frac{4!}{2^4 2!^2}$$

$$\frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)}{3!} = \frac{1\cdot 3\cdot 5}{2^3\, 3!} = \frac{6!}{2^6 (3!)^2}$$

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}(2xt-t^2)^n$$

↓factor out t^n

$$= \sum_{n=0}^{\infty} \frac{(2n)!\,t^n}{2^{2n}(n!)^2} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!}(2x)^{n-k} t^k$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(2n)!\,(-1)^k (2x)^{n-k} t^k t^n}{2^{2n} n!\, k!(n-k)!}$$

$$= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{[n/2]} \frac{(2n-2k)!}{2^{2n-2k} k!(n-k)!(n-2k)!}(2x)^{n-2k}$$

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(2n-2k)!\,(2x)^{n-2k}}{2^{2n-2k} k!(n-k)!(n-2k)!}$$

But

$$(x^2-1)^n = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} x^{2n-2k}$$

$$\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{2^n n!}\frac{(-1)^k n!}{k!(n-k)!}\frac{(2n-2k)!}{(n-2k)!} x^{n-2k}$$

$$= P_n(x)$$

Now orthogonality of P_ℓ^m follows just like P_ℓ's from d.e. Getting normalization is a little harder, I will just give result $m \geq 0$

$$\int_{-1}^{1} dx\, P_\ell^m(x) P_{\ell'}^m(x) = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'} \quad *$$

For $0 \leq m \leq \ell$

$$Y_\ell^m(\theta,\phi) = \left[\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}\right]^{1/2}(-1)^m e^{im\phi} P_\ell^m(\cos\theta)$$

$$Y_\ell^{-m} = (-1)^m (Y_\ell^m)^*$$

Solution To Rotationally Invariant Problems

Particle of mass μ moving in spherically sym potential $V(r)$

$$\left[\frac{-\hbar^2}{2\mu}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\right) + V(r)\right]\psi_E(r,\theta,\phi) = E\psi_E(r,\theta,\phi)$$

Seek energy eigenfunctions that are also eigenfunctions of L^2, L_z

$$\psi_{E\ell m}(r,\theta,\phi) = R_{E\ell m}(r)\, Y_\ell^m(\theta,\phi)$$

$$\left[-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right] + V(r)\right] R_{E\ell}(r) = E R_{E\ell}(r)$$

Note can drop subscript m since radial wf's don't depend on m. We have a $2\ell+1$ degeneracy for each ℓ

Convenient to rewrite in terms of $U_{E\ell}$ def. :

$$R_{E\ell} = \frac{U_{E\ell}}{r}$$

$$\frac{\partial}{\partial r} R_{E\ell r} = -\frac{1}{r^2} U_{E\ell}(r) + \frac{U'_{E\ell}}{r}$$

$$r^2 \frac{\partial}{\partial r} R_{E\ell r} = -U_{E\ell}(r) + r U'_{E\ell}(r)$$

$$\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} R_{E\ell}(r) = -U'_{E\ell}(r) + U'_{E\ell}(r) + r U''_{E\ell}(r)$$

$$\left\{-\frac{\hbar^2}{2\mu}\left[\frac{d^2}{dr^2} \qquad - \frac{\ell(\ell+1)}{r^2}\right] + V(r)\right\} U_{E\ell r} = E U_{E\ell}(r)$$

$$\left\{\frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V(r) - \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right]\right\} U_{E\ell}(r) = 0$$

Looks like 1-dim shrodinger eq. except

(i) Coordinate variable r goes from $0 \to \infty$ instead of $-\infty$ to ∞
(ii) In addition to potential a repulsive centrifugal barrier $\ell(\ell+1)\hbar^2/2\mu r^2$ in all but $\ell=0$ states
(iii) Boundary conditions on U different than 1-d case

Now lets consider what b.c.'s are. Let

$$D_\ell(r) = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V(r) + \frac{\ell(\ell+1)\hbar^2}{2mr^2}$$

$D_\ell(r)\,U_{E\ell}(r) = E\,U_{E\ell}(r)$. Hermiticity

$$\int_0^\infty U_1^*(D_\ell U_2)\,dr = \left[\int_0^\infty dr\; U_2^*(D_\ell U_1)\,dr\right]^* = \int_0^\infty dr\; U_2(D_\ell U_1)^*$$

But $\displaystyle\int_0^\infty U_1^*\frac{d^2}{dr^2}U_2\,dr$

$$= U_1^*\frac{dU_2}{dr}\Big|_0^\infty - \int_0^\infty \frac{dU_1^*}{dr}\frac{dU_2}{dr}$$

$$= U_1^*\frac{dU_2}{dr}\Big|_0^\infty - U_2\frac{dU_1^*}{dr}\Big|_0^\infty + \int_0^\infty \frac{d^2U_1^*}{dr^2}U_2$$

So need $\displaystyle\left(U_1^*\frac{dU_2}{dr} - U_2\frac{dU_1^*}{dr}\right)\Big|_0^\infty = 0$

Normalizable

$$\int_0^\infty r^2 dr\,|R_{E\ell}^2(r)| = \int_0^\infty dr\,|U_{E\ell}(r)|^2$$

$U_{E\ell}(r) \to 0$ as $r \to \infty$ for normalizable ← bound states

$r \to \infty$ for δ' function

Now for bound states ∞ doesn't contribute (also for states of same energy in unnormalizable δ-function case)

$$\left(U_1^*\frac{dU_2}{dr} - U_2\frac{dU_1^*}{dr}\right)\Big|_{r=0} = 0$$

$$U(r) \xrightarrow[r\to\infty]{} C \Rightarrow R \sim \frac{U}{r} \sim \frac{C}{r}$$

So the wf diverges. Not a problem R still square integrable. But it actually doesn't satisfy the Schrodinger eq $\psi \sim c/r$ $\psi_0^0 \sim c/r$

$$\nabla^2(1/r) = 4\pi\delta^3(\vec{r})$$

need this to be ...

So if $V(r)$ doesn't contain a δ-function at origin must have

$$U(r) \to 0 \quad \text{as } r \to 0$$

We can derive more properties by examining the equation it satisfies without using explicit form of the potential $V(r)$. Consider limit $r \to 0$, and assume $V(r)$ is less singular than r^{-2}. Then for small r equation is dominated by the centrifugal barrier

$$U_\ell'' \simeq \frac{\ell(\ell+1)}{r^2} u_\ell$$

We have dropped subscript E since it is inconsequential for dominant behavior near $r \to 0$. Try soln

$$U_\ell \sim r^\alpha$$

$$\alpha(\alpha-1) = \ell(\ell+1)$$

$$\alpha = \ell+1 \quad \text{or} \quad \alpha = -\ell$$

$$U_\ell \sim \begin{cases} r^{\ell+1} & \text{regular soln} \\ r^{-\ell} & \text{irregular soln} \end{cases}$$

$$d_\ell^+ d_\ell (d_\ell^+ U_\ell) = d_\ell^+ U_\ell$$

$$d_{\ell+1} d_{\ell+1}^+ (d_\ell^+ U_\ell) = d_\ell^+ U_\ell$$

So

$$d_\ell^+ U_\ell = c_\ell U_{\ell+1}$$

where c_ℓ is a constant. If $\ell = 0$ then there are two solutions

$$U_0^A(\rho) = \sin\rho \quad , \quad U_0^B(\rho) = \cos\rho$$

If origin included in region considered then $U_0^B(\rho)$ is forbidden. Now lets see explicitly how other solns are built

$$\rho R_{\ell+1} = d_\ell^+ \rho R_\ell$$

$$= \left(\frac{-d}{d\rho} + \frac{\ell+1}{\rho}\right)(\rho R_\ell)$$

$$\Rightarrow R_{\ell+1} = \left(-\frac{d}{d\rho} + \frac{\ell}{\rho}\right) R_\ell$$

$$\Rightarrow R_{\ell+1} = \rho^\ell \left(-\frac{d}{d\rho}\right)\frac{R_\ell}{\rho^\ell}$$

Now then

$$\frac{R_{\ell+1}}{\rho^{\ell+1}} = -\frac{1}{\rho}\frac{d}{d\rho}\frac{R_\ell}{\rho^\ell} = \left(-\frac{1}{\rho}\frac{d}{d\rho}\right)^2 \frac{R_{\ell-1}}{\rho^{\ell-1}}$$

$$= \left(-\frac{1}{\rho}\frac{d}{d\rho}\right)^{\ell+1} \frac{R_0}{\cancel{\rho^0}\,1}$$

← overall minus sign from above

$$\Rightarrow R_\ell = (-\rho)^\ell \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^\ell R_0$$

$$R_0^A = \sin\rho/\rho \quad , \quad R_0^B = -\cos\rho/\rho$$

Spherical Bessel Functions

$$R_\ell^A \equiv j_\ell = (-\rho)^\ell \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^\ell \left(\frac{\sin\rho}{\rho}\right)$$

Spherical Neuman Functions of order ℓ

$$R_\ell^B \equiv n_\ell = (-\rho)^\ell \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^\ell \left(\frac{-\cos\rho}{\rho}\right)$$

$$j_0(\rho) = \frac{\sin\rho}{\rho} \qquad n_0(\rho) = \frac{-\cos\rho}{\rho}$$

$$j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho} \qquad n_1(\rho) = -\frac{\cos\rho}{\rho} - \frac{\sin\rho}{\rho}$$

Can show as $\rho \to \infty$ these behave

$$j_\ell \to \frac{1}{\rho}\sin\left(\rho - \frac{\ell\pi}{2}\right)$$

$$n_\ell \to \frac{1}{\rho}\cos\left(\rho - \frac{\ell\pi}{2}\right)$$

$j_\ell(\rho)$ finite as $\rho \to 0$

$$j_\ell(\rho) \xrightarrow[\rho\to 0]{} \frac{\rho^\ell}{(2\ell+1)!!}$$

$$(2\ell+1)!! = (2\ell+1)(2\ell-1)\cdots(5)(3)(1).$$

But

$$n_\ell(\rho) \xrightarrow[\rho\to 0]{} \frac{-(2\ell-1)!!}{\rho^{\ell+1}}$$

Free particle wf's which are regular at small r (?)

$$\psi_{Elm}(r,\theta,\phi) = j_l(kr)\, Y_l^m(\theta,\phi)$$

Connection to Soln in Cartesian Coord

$$\psi_E(x,y,z) = \frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{p}\cdot\vec{r}/\hbar} \qquad E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

$$\vec{p} = \hbar\vec{k}$$

$$\frac{\vec{p}\cdot\vec{r}}{\hbar} = kr\cos\theta$$

$$\psi_E(r,\theta,\phi) = \frac{e^{ikr\cos\theta}}{(2\pi\hbar)^{3/2}}$$

Now can expand in spherical Harmonics

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_l^m\, j_l(kr)\, Y_l^m(\theta,\phi)$$

LHS independent of m so only m=0 contributes

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} C_l^0\, j_l(kr)\, Y_l^0(\theta,\phi)$$

$$Y_l^0 = \left(\frac{2l+1}{4\pi}\right)^{1/2} P_l(\cos\theta)$$

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} C_l\, j_l(kr)\, P_l(\cos\theta)$$

Can show $C_l = i^l(2l+1)$

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l(2l+1)\, P_l(\cos\theta)\, j_l(kr)$$

Free Particle in Spherical Coordinates

Whole wf

$$\psi_{E\ell m}(r,\theta,\phi) = R_{E\ell}(r)\,Y_{\ell m}(\theta,\phi)$$

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2}\right] U_{E\ell}(r) = 0 \qquad k^2 = \frac{2mE}{\hbar^2} \quad \text{mass}$$

Dividing by k^2 & introducing $\rho = kr$

$$\left[-\frac{d^2}{d\rho^2} + \frac{\ell(\ell+1)}{\rho^2}\right] U_\ell = U_\ell$$

Define operators analogous to raising & lowering operators

$$d_\ell = \frac{d}{d\rho} + \frac{\ell+1}{\rho}, \qquad d_\ell^+ = -\frac{d}{d\rho} + \frac{\ell+1}{\rho}$$

The differential equation is

$$d_\ell d_\ell^+ U_\ell = U_\ell \qquad *$$

Now: $d_{\ell+1} d_{\ell+1}^+ U = \left(\frac{d}{d\rho} + \frac{\ell+2}{\rho}\right)\left(-\frac{d}{d\rho} + \frac{\ell+2}{\rho}\right) U$

$$= -\frac{d^2}{d\rho^2} U + \left(\frac{\ell+2}{\rho}\right)\left(\frac{\ell+2}{\rho}\right) U + \left[\frac{d}{d\rho}, \frac{\ell+2}{\rho}\right] U$$

$$= -\frac{d^2}{d\rho^2} U + \frac{(\ell+2)}{\rho}\frac{(\ell+2)}{\rho} U - \frac{\ell+2}{\rho^2} U$$

$$= -\frac{d^2}{d\rho^2} + \frac{\ell^2 + 4\ell + 4 - \ell - 2}{\rho^2} U = d_\ell^+ d_\ell U$$

$\Rightarrow d_{\ell+1} d_{\ell+1}^+ = d_\ell^+ d_\ell$

In * multiply both sides by d_ℓ^+

We reject irregular soln since it doesn't satisfy condition $U_{E,l}(r) \to 0$ as $r \to 0$. Note this only holds for $l > 0$. For $l = 0$ centrifugal barrier is irrelevant. Although $U_0(r) \to 0$ as $r \to 0$ particle has non zero amplitude to be at the origin because $R_0(r) = U_0(r)/r \neq 0$ at $r = 0$. For most problem we consider $U_0(r) \sim r$ as $r \to 0$ so behavior even holds for $l = 0$.

Next consider the behavior of $U_{E,l}(r)$ as $r \to \infty$. If $V(r)$ does not vanish as $r \to \infty$ it will dominate the result. So let's consider case $rV(r) \to 0$ as $r \to \infty$. At large r eq becomes

$$\frac{d^2 U_{El}}{dr^2} = -\frac{2mE}{\hbar^2} U_E$$

1) $E > 0$ the particle is allowed to escape to ∞ classically. We expect $U_{El}(r)$ to oscillate as $r \to \infty$

2) $E < 0$ the particle is bound. The region $r \to \infty$ is forbidden classically and we expect $U_{El}(r)$ to fall off exponentially there

First case $E > 0$

$$U_E = Ae^{ikr} + Be^{-ikr} \qquad k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$$

Particle behaves as a free particle far from origin. Let's write soln in this case for finite r

$$U_E = f(r)e^{\pm ikr}$$

and ignore centrifugal barrier

$$f'' \pm (2ik)f' - \frac{2m}{\hbar^2}V(r)f = 0$$

We expect for very large r that f(r) is slowly varying so let's ignore $f''(r)$

$$\frac{df}{f} = \mp \frac{i}{k}\frac{\mu}{\hbar^2} V(r)\,dr$$

$$f(r) = f(r_0)\exp\left[\mp \frac{i\mu}{k\hbar^2}\int_{r_0} V(r)\,dr\right]$$

where r_0 is some constant. If $rV(r) \to 0$ as $r \to \infty$ can do integral as $r \to \infty$ it approaches a constant given behavior we want. But if instead

$$V(r) = \frac{-e^2}{r}$$

$$f(r) = f(r_0)\exp\left[\frac{\pm i\mu e^2}{k\hbar^2}\ln\left(\frac{r}{r_0}\right)\right]$$

$$U_E(r) \sim \exp\left[\pm i\left(kr + \frac{\mu e^2}{k\hbar^2}\ln r\right)\right]$$

If $V(r)$ falls slower than $1/r$ problem is even worse.

Next consider case $E<0$. All the results with $E>0$ carry over with change

$$k \to i\kappa \qquad \kappa = \left(\frac{2\mu|E|}{\hbar^2}\right)^{1/2}$$

$$U_E \xrightarrow[r\to\infty]{} Ae^{-\kappa r} + Be^{\kappa r}$$

A/B is not arbitrary since U_E continued inward to $r \to 0$ must vanish. The growing exponential must vanish for normalizability. So there only get soln for certain discrete values of E. (If A/B were arbitrary just set $B=0$ without constraining E)

Note as before also analyze $rV(r) \to 0$ as $r \to \infty$
As before and $k \to i\kappa$

$$u_E(r) \sim \exp\left[-\kappa r\right] \exp\left[\frac{+\mu e^2}{\kappa \hbar^2} \ln r\right]$$

$$\sim r^{(\mu e^2/\kappa \hbar^2)} e^{-\kappa r}$$

When $E<0$ eigenfunctions are normalizable to unit

$$\int_0^\infty u_{E'\ell}(r)\, u_{E\ell}(r) = \delta_{EE'}$$

$$\psi_{E\ell m}(r,\theta,\phi) = R_{E\ell}(r)\, Y_{\ell m}(\theta,\phi)$$

$$\iiint \psi^*_{E\ell m}(r,\theta,\phi)\, \psi_{E'\ell' m'}(r,\theta,\phi)\, r^2 d\Omega$$

$$= \delta_{EE'}\delta_{\ell\ell'}\delta_{mm'}$$