Phys 125
Quantum Mechanics

Text: R. Shankar, Principles of Quantum Mechanics, second edition, Springer (1994)

Chapter 1: Mathematical Review

Definition: <u>Vector space</u> $\mathbb{V}$ is a collection objects $|1\rangle, |2\rangle, \ldots$ $|V\rangle, \ldots |W\rangle, \ldots$ called vectors for which

1. There is a rule for adding vectors $|V\rangle + |W\rangle$
2. There is a rule for multiplying by scalars $a, b \ldots$ $a|V\rangle$

$\hookrightarrow$ either complex or real numbers

Such that

(i) The result of vector addition or scalar multiplication is another vector $|V\rangle + |W\rangle \in \mathbb{V}$, $a|V\rangle \in \mathbb{V}$.

(ii) Scalar multiplication is distributive over vector addition $a(|V\rangle + |W\rangle) = a|V\rangle + b|W\rangle$

(ii) Scalar multiplication is distributive over scalar addition $(a+b)|V\rangle = a|V\rangle + b|V\rangle$

(iii) Scalar multiplication is associative $a(b|V\rangle) = (ab)|V\rangle$

(iv) Vector addition is associative $|V\rangle + (|W\rangle + |Z\rangle) = (|V\rangle + |W\rangle) + |Z\rangle$

(v) Vector addition is commutative $|V\rangle + |W\rangle = |W\rangle + |V\rangle$

(vi) $\exists$ a null vector $|0\rangle$ obeying $|0\rangle + |V\rangle = |V\rangle$

(vii) For every vector $|V\rangle$ $\exists$ an inverse $|-V\rangle$ such that $|V\rangle + |-V\rangle = |0\rangle$

Some properties of vector space

$|0\rangle$ is unique

$|0\rangle = 0|V\rangle$ for any $|V\rangle$

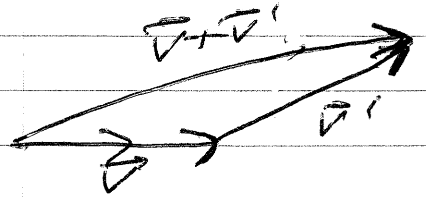
$|-V\rangle = -|V\rangle$

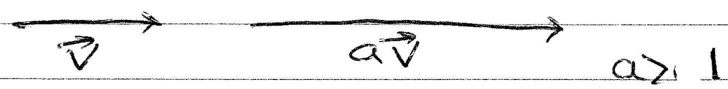
Inverse of a vector is unique

Eg. of vector space

(a) Ordered 3-tuples of real numbers (a, b, c) with vector addition defined by $(a, b, c) + (e, f, g) = (a+e, b+f, c+g)$ and scalar multiplication $g(a, b, c) = (ga, gb, gc)$. Null vector $|0\rangle = (0,0,0)$.

– Note this vector space has geometric interpretation. Let (a, b, c) be 3 coordinates of a pt in 3 dimensional space & the vector $\vec{V} = a\hat{x} + b\hat{y} + c\hat{z}$, where $\hat{x}, \hat{y}, \hat{z}$ are unit vectors along x, y, z axis. For this vector space





(b). 2x2 Matrices $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$

Well these are really ordered 4-tuples so pretty obvious they form a vector space

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & +d+h \end{pmatrix} \Leftarrow \text{vector addition}$$

$$x \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} xe & xf \\ xg & xh \end{pmatrix} \Leftarrow \text{scalar multiplication}$$

- If scalars a are real numbers call a real vector space if complex numbers a complex vector space

- Defn Linear Independence

Consider a relation of form

$$\sum_{i=1}^{n} a_i |i\rangle = |0\rangle$$

assume none of vectors on lhs. are $|0\rangle$.

- A set of vectors is linearly independent if only solution to above eq. has all $a_i = 0$. If set of vectors is not linearly independent then they are called *linearly independent*.

For linearly independent vectors it is not possible to write one of the vectors in terms of the others. On the other hand if the vectors are *linearly dependent*

then one of the a_i's is non zero. Say its $i=1$.
Then

$$|1\rangle = \sum_{i=2}^{n} \frac{-a_i}{a_1} |i\rangle,$$

expressing $|1\rangle$ in terms of the others

Definition: Dimension: A vector space has dimension n if it can have a maximum of n linearly independent vectors. Denote $V^n(R)$ if it is real or $V^n(C)$ if it is complex.

- Theorem. In an n-dimensional vector space any vector can be written as a linear combination of n linearly independent vectors.

Proof: Suppose a vector $|V\rangle$ for which this is not possible. It would join set of n linearly independent vectors to form a set of $n+1$ linearly independent vectors

Defn. Basis: A set of n-linearly independent vectors in an n-dimensional vector space is called a basis. For any vector can write

$$|V\rangle = \sum_{i=1}^{n} v_i |i\rangle$$

v_i are called the components.

This expansion is unique. Suppo

$$|V\rangle = \sum_{i=1}^{n} v_i' |i\rangle$$

$$|0\rangle = \sum_{i=1}^{n} (v_i - v_i')|i\rangle$$

Since $|i\rangle$ are linearly independent $v_i = v_i'$.

Note adding vectors amounts to adding their components

$$|V\rangle = \sum_i v_i|i\rangle, \quad |W\rangle = \sum_i w_i|i\rangle$$

$$|V\rangle + |W\rangle = \sum_i (v_i + w_i)|i\rangle$$

and scalar multiplication amounts to multiplying the components by the scalar

$$a|V\rangle = \sum_i (av_i)|i\rangle$$

For usual 3-dimensional vectors $\vec{A}, \vec{B}$ define inner product $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\Theta$ want to generalize this. Note inner product takes two vectors $\vec{A}, \vec{B}$ & produces a real number or scalar $\vec{A}\cdot\vec{B} = f(\vec{A},\vec{B})$. Properties $f(\vec{A},\vec{B}) = f(\vec{B},\vec{A})$
$f(\alpha\vec{A} + \gamma\vec{C}, \vec{B}) = \alpha f(A, \vec{B}) + \beta f(\vec{C}, \vec{B})$.
$f(\vec{A},\vec{A}) \geq 0$, $f(\vec{A},\vec{A}) = \vec{0} \Rightarrow |\vec{A}\rangle = \vec{0}$

Denote inner (or scalar) product of two vectors $|V\rangle, |W\rangle$ by $\langle V|W\rangle$. It has properties

$\langle V|W\rangle = \langle W|V\rangle^*$ ← scalars complex (Skew symmetry)
$\langle V|V\rangle \geq 0$ with equality iff $|V\rangle = |0\rangle$.
$\langle V|(a|W\rangle + b|Z\rangle) \equiv \langle V|aW + bZ\rangle = a\langle V|W\rangle + b\langle V|Z\rangle$

Vector space with inner product defined called an inner product space

Note

$$\langle aW+bZ|V\rangle = \langle V|aW+bZ\rangle^*$$

$$= a^*\langle V|W\rangle^* + b^*\langle V|Z\rangle$$
$$= a^*\langle W|V\rangle + b^*\langle Z|V\rangle$$

Antilinearity in first factor.

Defn. Orthogonal: Two vectors are orthogonal or perpendicular if their inner product vanishes

Call $\sqrt{\langle V|V\rangle} = |V|$ the norm of vector. A normalized vector has unit norm.

A set of n linearly independent unit norm vectors that are pairwise orthogonal is called an o.n. basis.

Given $|V\rangle$, $|W\rangle$

$$|V\rangle = \sum_i v_i |i\rangle \qquad |W\rangle = \sum_j w_j |j\rangle$$

↳ basis

$$\langle V|W\rangle = \sum_{i,j} v_i^* w_j \langle i|j\rangle$$

Suppose basis o.n. $\langle i|j\rangle = \delta_{ij}$ → Kronecker delta
For an o.n. basis

$$\langle V|W\rangle = \sum_i v_i^* w_i$$

$$\langle V|V\rangle = \sum_i |v_i|^2 \geq 0$$

For o.n. basis $|V\rangle = \sum_i |i\rangle\langle i|V\rangle$

There is a procedure to take any basis & convert it to an o.n one. It is called Gram-Schmidt procedure. Let

$|I\rangle, \; |II\rangle, \; |III\rangle, \ldots$

be a basis. Then

$$|1\rangle = \frac{|I\rangle}{|I|}$$

Then

$$|2'\rangle = |II\rangle - |1\rangle\langle 1|II\rangle$$

$$\langle 1|2'\rangle = \{\langle 1|II\rangle - \langle 1|II\rangle\} = 0$$

$$|2\rangle = \frac{|2'\rangle}{|2'|}$$

$$|3'\rangle = |III\rangle - |1\rangle\langle 1|III\rangle - |2\rangle\langle 2|III\rangle$$

Note $|3'\rangle$ is orthogonal to $|1\rangle, |2\rangle$

$$\langle 1|3'\rangle = \langle 1|III\rangle - \langle 1|III\rangle = 0$$ + sim for 2

$$|3\rangle = \frac{|3'\rangle}{|3|}$$

& keep going. Note would terminate if got zero but cannot because then would have a nontrivial l.c of a vectors that adds to zero

l.i

Example

$$|I\rangle = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad |II\rangle = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad |III\rangle = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|2'\rangle = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} 0$$

$$|2\rangle = \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$|3'\rangle = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} - |1\rangle 0 - |2\rangle \left(\frac{2}{\sqrt{5}} + \frac{10}{\sqrt{5}} \right)$$

$$= \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 12/5 \\ 24/5 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/5 \\ +1/5 \end{bmatrix}$$

$$|3\rangle = \begin{bmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \qquad \| |3'\rangle \| = 1/\sqrt{5}$$

We have been using $|V\rangle$ for vector this is called ket notation in innerproduct $\langle W|V\rangle$ it can be convenient to think of $\langle W|$ as a real object. It is called a bra vector in the space dual to the original vector space. What bra corresponding to vector $a|W\rangle$ call this bra $|\tilde{W}\rangle$

$$\langle \tilde{W}|V\rangle = \langle aW|V\rangle = a^*\langle W|V\rangle$$

Write $\langle aW| = \langle W|a^*$

$$\langle aW|V\rangle = \langle W|a^*V\rangle$$

Can think of th

$$|V\rangle = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{in o.n bas}$$

$$\langle V| = [v_1^*, \ldots\ v_n^*]$$

$$a|V\rangle = b|W\rangle + c|Z\rangle$$

$$\Rightarrow \langle V|a^* = \langle W|b^* + \langle Z|c^*$$

Given $|V\rangle = \sum_i v_i|i\rangle$

$$\langle V| = \sum_i \langle i|v_i^*$$

$$\langle V| = \sum_i \langle V|i\rangle\langle i| \rightarrow \text{expand in O.n basis}$$

Two very important inequalities

SCHWARZ INEQUALITY $|\langle V|W\rangle| \leq |V||W|$

TRIANGLE INEQUALITY $|V+W| \leq |V|+|W|$

Easy to prove. For Schwarz consider

$$|Z\rangle = |V\rangle - \frac{\langle W|V\rangle}{|W|^2}|W\rangle$$

$$\langle Z|Z\rangle = \left(\langle V| - \langle W|\frac{\langle V|W\rangle}{|W|^2}\right)\left(|V\rangle - \frac{\langle W|V\rangle}{|W|^2}|W\rangle\right)$$

$$= \langle V|V\rangle - \frac{|\langle V|W\rangle|^2}{|W|^2} - \frac{|\langle V|W\rangle|^2}{|W|^2} + \frac{\cancel{\langle W|W\rangle}}{\cancel{|W|^4}}|\langle V|W\rangle|^2$$

$1/|W|^2$

≥ 0

$$|V|^2|W|^2 \geq |\langle V|W\rangle|^2 \quad \checkmark$$

For triangle

$$|V+W|^2 = \langle V|V\rangle + \langle V|W\rangle + \langle W|V\rangle + \langle W|W\rangle$$

$$= |V|^2 + |W|^2 + 2\,\mathrm{Re}\,\langle V|W\rangle$$

$$\leq |V|^2 + |W|^2 + 2|\langle V|W\rangle|$$

$$\leq |V|^2 + |W|^2 + 2|V||W| \qquad \text{Swartz}$$

$$\leq (|V| + |W|)^2$$

Definition: Subspace: Given a vector space $\mathbb{V}$, a subset of its elements that form a vector space among themselves (with vector addition + scalar multiplication defined as in $\mathbb{V}$) is called a vector subspace. A vector subspace i of dimensionality n_i is denoted $\mathbb{V}_i^{n_i}$.

Definition: Direct Sum of Vector Subspaces: Given two subspaces $\mathbb{V}_i^{n_i}$, $\mathbb{V}_j^{n_j}$ we define the sum $\mathbb{V}_k^{m_k} = \mathbb{V}_i^{n_i} \oplus \mathbb{V}_j^{n_j}$ as the set of vectors contained in the subspaces and all linear combinations of those.

Linear Operators

An operator $|\Omega\rangle$ is an instruction for taking a vector $|V\rangle$ into another $|V'\rangle$. The action of operators is written

$$\Omega|V\rangle = |V'\rangle$$

The operator Ω transformed the ket $|V\rangle$ into the ket $|V'\rangle$. Similarly for the bras

$$\langle W|\Omega = \langle W'|$$

In quantum mechanics we need linear operators

$$\Omega\{\alpha|V\rangle + \beta|W\rangle\} = \alpha\,\Omega|V\rangle + \beta\,\Omega|W\rangle$$

Similarly for "bras". One trivial operator is the identity $I|V\rangle = |V\rangle$, $\langle W|I = \langle W|$

An 3x3 matrix is a linear operator acting on the vector space of 3-tuples in the usual way

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 v_1 + a_2 v_2 + a_3 v_3 \\ b_1 v_1 + b_2 v_2 + b_3 v_3 \\ c_1 v_1 + c_2 v_2 + c_3 v_3 \end{pmatrix}$$

If the 3x3 matrix is orthogonal then it is a rotation of the vector

A really neat property of linear operators is that once their action on basis vectors is known then action on whole vector space is determined: If

$$\Omega|i\rangle = |i'\rangle$$

for any basis $|1\rangle, |2\rangle \ldots |n\rangle$ of $\mathbb{V}^n$ then for any $|V\rangle = \sum_i v_i |i\rangle$

$$\Omega|V\rangle = \sum_i v_i \, \Omega|i\rangle = \sum_i v_i |i'\rangle$$

Product of two linear operators Λ, Ω is defined

$$\Lambda\Omega|V\rangle = \Lambda\left(\Omega|V\rangle\right) = \Lambda|\Omega V\rangle$$

at $|\Omega V\rangle$ is the ket obtained when Ω acts on $|V\rangle$