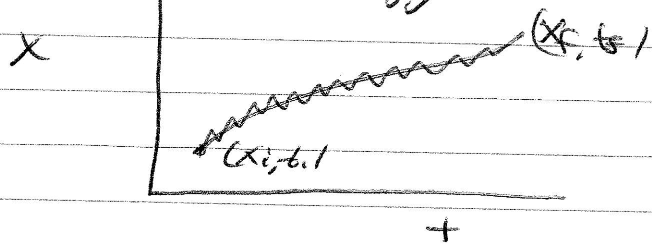
Least Action it isn't

We found classical equations of motion by demanding action be stationary. But is the classical path a minimum. There are always neighbouring paths that increase the classical action. For example wiggly path below has almost same potential at each pt but much higher kinetic energy.



So the classical path is not a local maximum. But it isn't always a local minimum either. As an example consider the harmonic oscillator

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2$$

with $m = k = 1$. Equation of motion

$$\ddot{x} + x = 0$$

Soln $x = x_0 \cos t$ for classical path that has $x = x_0$ at $t = 0$ & $x = x_0$ at $t = 2\pi$.
This is a path that begins & ends at x_0. Consider neighbouring path

$$x(t) = (1-\alpha)\, x_0 \cos t + \alpha x_0$$

For some constant α. Plug this in Lagrange & do integral to get action

$$S[\alpha] = \int_0^{2\pi} dt \left[\cancelto{\sim}{(1-\alpha)^2 x_0^2} \sin^2 t - \frac{1}{2} \cancelto{\sim}{(1-\alpha)^2} x_0^2 \cos^2 t - \frac{2\alpha x_0^2}{2} \cancelto{0}{\cos t} - \frac{1}{2}\alpha^2 x_0^2 \right]$$

$$= -\pi x_0^2 \alpha^2$$

$\alpha = 0$ is classical path & any non zero α lowers the action. So classical path is a saddle pt in this case not a local minimum.