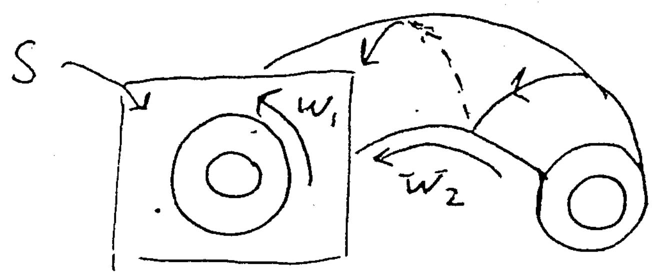
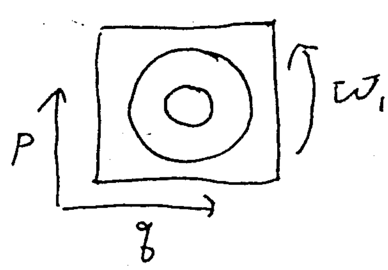
Perturbing the Twist map

Continue to consider the $N=2$ case. We'll examine in more detail what happens to the resonant tori when an integrable system is perturbed.



Suppose we have found action-angle variables for the integrable system: J_1, J_2, w_1, w_2

We can choose the surface S that is used to construct the Poincaré map to be the surface $w_2 = 0$. Then we may use as coordinates on S the variables w_1, J_1 (J_2 is fixed by the condition $H(J_1, J_2) = E$, once we specify J_1. So in the space $H=E$, there are <u>nested</u> <u>tori</u> labeled by J_1)



In S, the tori become concentric circles.

$\theta = 2\pi w_1$ is the angular coordinate in the plane

We may choose J_1 to be the "radial coordinate" --

$$J_1 = \oint_1 p\, dq \longrightarrow 0$$ at the "origin", where the circle shrinks to zero

or -- if we want ρ to be the usual radial coordinate, with $\pi\rho^2$ being the area of circle of radius ρ:

Define ρ by $\quad J_1 = \oint p\,dq = \text{area} = \pi\rho^2$

What is the Poincaré map (for the integrable system)? In angle variables, the trajectories are..

$$w_1 = \nu_1 t + \text{const},$$
$$w_2 = \nu_2 t + \text{const}.$$

so the orbit returns to $w_2 = 0$ after

$$\Delta t = \nu_2^{-1} \quad \text{elapses}$$

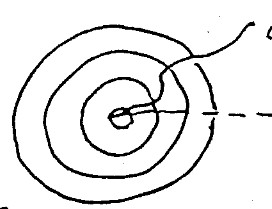
Thus

$$\Delta w_1 = \frac{\nu_1}{\nu_2} \equiv \alpha(J_1) = \alpha(\rho)$$

or

$$\boxed{\Delta\theta = 2\pi\,\alpha(\rho)}$$

This is a "Twist map" — each circle (of constant radius ρ) is rigidly rotated by the angle $2\pi\alpha$ — but with angle of rotation depending on the radius ρ



The KAM theorem can be expressed as a statement about perturbations of the twist map

The unperturbed twist map T is

$$T: \quad \rho_{n+1} = \rho_n$$
$$\theta_{n+1} = \theta_n + 2\pi\,\alpha(\rho_n)$$

A generic small perturbation of the map has the form

$$T_\epsilon : \rho_{n+1} = \rho_n + \epsilon f(\rho_n, \theta_n)$$
$$\theta_{n+1} = \theta_n + 2\pi\alpha(\rho_n) + \epsilon g(\rho_n, \theta_n)$$

KAM say that, for $\epsilon \ll 1$, most points in the plane lie on invariant curves of the perturbed map. Only the circles for which α is "sufficiently close to rational" are in danger of being destroyed. This special ($N=2$) case of the KAM theorem applies to any sufficiently smooth area preserving perturbation.

Resonant tori of the unperturbed system are those for which $\alpha(\rho)$ = rational number. Suppose

$$\alpha(\rho_0) = \frac{r}{s}$$ where r and s are integers that are relatively prime (no common factor)

Then all points on the circle of radius ρ_0 lie on finite periodic orbits of the twist map T. Points on the orbit are

$$\theta_0, \ \theta_0 + 2\pi\frac{r}{s}, \ \theta_0 + 2\pi\frac{2r}{s}, \ --- \ \theta_0 + 2\pi(s-1)\frac{r}{s}$$

– (orbit of length s)

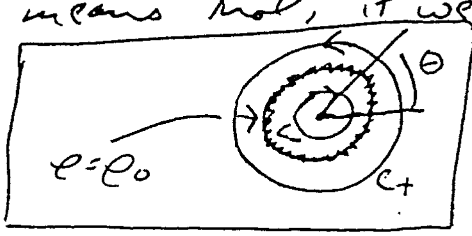
Every point on this circle is a fixed point of the map T^S -- which rotates the circle

Now imagine turning on the perturbation, and consider T_ϵ^S. What happens to the circle of fixed points of T^S when $\epsilon > 0$?

The generic perturbation removes most of these fixed points, but not all of them. In fact, we can show that, generically, the number of fixed points that survive is an even (integer) multiple of S.

(This is called the Poincaré-Birkhoff fixed point theorem.)

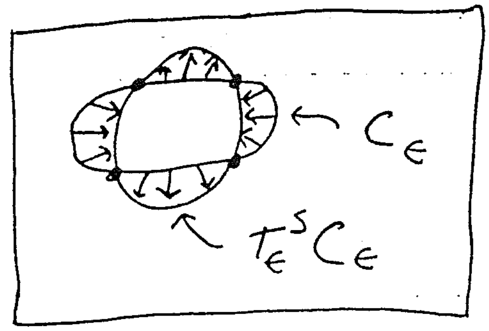
To see this, first consider the unperturbed map again. Generically $\alpha'(\rho) \neq 0$ at $\rho = \rho_0$. This means that, if we consider circles



C_+ with ρ slightly larger than ρ_0 and C_- with ρ slightly smaller, The map T^S (which leaves the $\rho = \rho_0$ circle fixed) rotates C_+ slightly one way and C_- slightly the other way. If ϵ is very small, this means that, for each value of the polar angle θ, there is a point between C_- and C_+ that is mapped radially by T_ϵ^S. Thus there is a closed curve C_ϵ that is mapped radially by T_ϵ^S, and approaches the $\rho = \rho_0$ circle as $\epsilon \to 0$.

Now, we remember that the T^S is [illegible] preserving

Therefore, the closed curves C_ϵ and $T_\epsilon^s C_\epsilon$ enclose the <u>same</u> <u>area</u>. This means that C_ϵ and $T_\epsilon^s C_\epsilon$ must <u>intersect</u>.



Generically (excluding the possibility that the two closed curves are tangent somewhere, which can happen only at isolated values of ϵ) the two curves must intersect in an <u>even</u> number of points. These points of intersection are the fixed points of T_ϵ^s.

In fact, the number of fixed points must be a multiple of s. Suppose X_0 is a fixed point of T_ϵ^s

Then $X_0, T_\epsilon X_0, T_\epsilon^2 X_0, \ldots T_\epsilon^{s-1} X_0$ are s distinct points — a closed orbit of T_ϵ that becomes the order s orbit of T as $\epsilon \to 0$. Each of these points is a fixed point

$$T_\epsilon^s(T_\epsilon^m X_0) = T_\epsilon^m T_\epsilon^s X_0 = T_\epsilon^m X_0$$

Since the number of fixed points is even and a multiple of s, it is an even multiple of s if s is odd. We will see momentarily that the number of fixed points must be an even multiple of s, even if s

Classification of Fixed Points

Next we want to consider the issue of the stability of the fixed points. If we imagine iterating T_ϵ^S many times, we want to know whether points close to the fixed point stay on orbits that remain close to the fixed point, or whether they are eventually driven away.

We study stability be linearizing the map near the fixed points. For a two-dimensional map

$$M: \begin{array}{l} x_1 \to M_1(x_1, x_2) \\ x_2 \to M_2(x_1, x_2) \end{array}$$

the linearized map at X_0 is the 2×2 matrix

$$\underset{\sim}{DM}(X_0) = \begin{pmatrix} \partial M_1/\partial x_1 & \partial M_1/\partial x_2 \\ \partial M_2/\partial x_1 & \partial M_2/\partial x_2 \end{pmatrix}\Bigg|_{X=X_0}$$

then $M: X_0 + \delta X \to M(X_0 + \delta X)$

$$= M(X_0) + \underset{\sim}{DM} \cdot \delta X = X_0 + \underset{\sim}{DM} \cdot \delta X$$

or $M: \delta X \to \underset{\sim}{DM}\, \delta X$

If M is an area-preserving map, then

$$\det(\underset{\sim}{DM}) = \pm 1$$

(-1 if the map is orientation-reversing).

(In fact, for the Poincare map of a Hamiltonian system, the linearized map has determinant 1 --- it is a symplectic matrix, as a consequence of the symplectic structure that is preserved by Hamiltonian time evolution.)

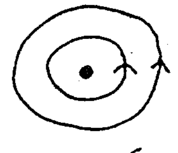
The linearized map has two eigenvalues, and the product of the eigenvalues is 1 (since $\det \underline{DM} = 1$).

Now distinguish two cases...

- Complex eigenvalues ($\mathrm{Im}\,\lambda \neq 0$)

The eigenvalues come in a complex conjugate pair $e^{i\beta}, e^{-i\beta}$.

There is a (not necessarily orthogonal) basis in which $\underline{DM}$ is a rotation by the angle β



In this basis, it is clear that the invariant curves preserved by iterations of $\underline{DM}$ are circles about the fixed point. Hence the fixed point is stable

In the original basis, we have

$$\underset{\sim}{DM} = \underset{\sim}{A}\ \underset{\sim}{R}(\beta)\ \underset{\sim}{A}^{-1}$$

where $\underset{\sim}{R}(\beta)$ is the rotation by β — So $\underset{\sim}{DM}$ differs from $\underset{\sim}{R}(\beta)$ by a similarity transformation.

Recall that a general linear transformation takes a circle to an ellipse:

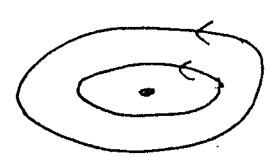
If $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, then

$R^2 = x^2 + y^2$ becomes $R^2 = (ax+by)^2 + (cx+dy)^2$

we can rediagonalize this quadratic form to obtain

$$1 = \frac{(x-c_1)^2}{a_1^2} + \frac{(y-c_2)^2}{a_2^2}$$

Thus, in general, the invariant curves of a linear transformation with eigenvalues $e^{\pm i\beta}$ are ellipses:



($\underset{\sim}{A}^{-1}$ takes ellipse to a circle, which is preserved by $R(\beta)$, and then is mapped back to the ellipse by $\underset{\sim}{A}$.)

For this reason, stable fixed points of an area preserving map are called "elliptic fixed points"

- Real Eigenvalues

The two eigenvalues are λ, λ^{-1}

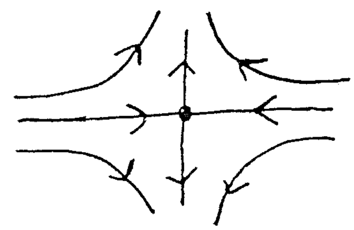
Suppose $|\lambda| > 1$

There is a (not necessarily orthogonal) basis in which transformation is

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

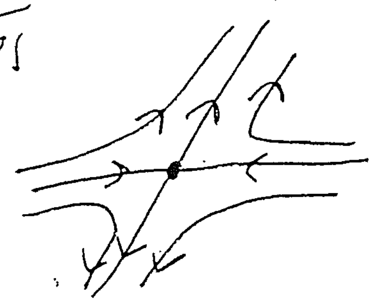
The invariant curves are hyperbolas

$xy = \text{constant}$



For $\underline{DM} = \underline{A} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \underline{A}^{-1}$, then,

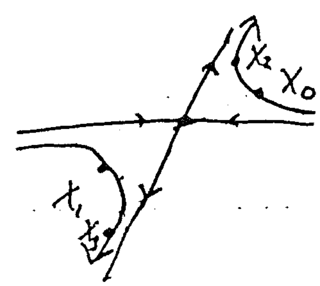
the invariant curves are still hyperbolas centered at the fixed point, because a linear transformation maps a hyperbola to a hyperbola — except the asymptotes need not be orthogonal.



This type of fixed point is called hyperbolic.

It is clearly unstable -- along the eigenvector with eigenvalue $|\lambda| > 1$, nearby points exponentially diverge from the fixed point, separation increasing like $|\lambda|^m$ after m iterations.

(There are two types of hyperbolic fixed point: either $\lambda > 1$ or $\lambda < -1$. In the latter case, orbit hops back and forth between the two branches of the hyperbola. We then speak of a "hyperbolic fixed point with reflection.")



On the boundary between the elliptic and hyperbolic cases is the case $\lambda_1 = \lambda_2 = \pm 1$. Then there is a basis in which

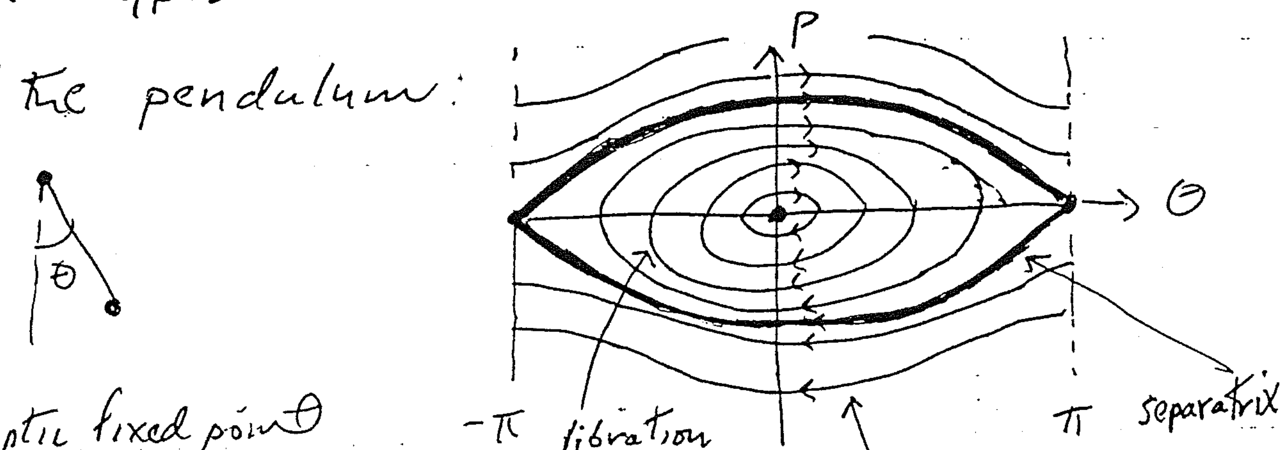
$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (\text{if } \lambda = +1)$$

The invariant curves are the lines $y =$ constant (with $y = 0$ corresponding to a <u>line of fixed points</u>).

In this case we say the fixed point is <u>parabolic</u>. Parabolic fixed points are <u>nongeneric</u> and become elliptic or hyperbolic when slightly perturbed.

Examples of fixed points of all three types occur in $N=1$ flow diagrams

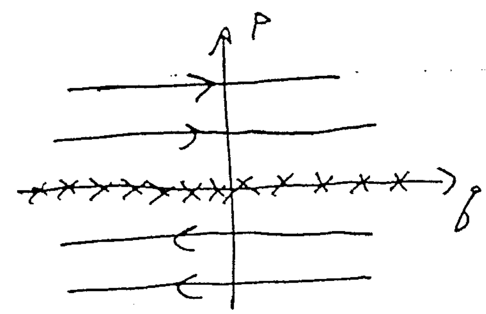
For the pendulum:



Elliptic fixed point at $p = \theta = 0$

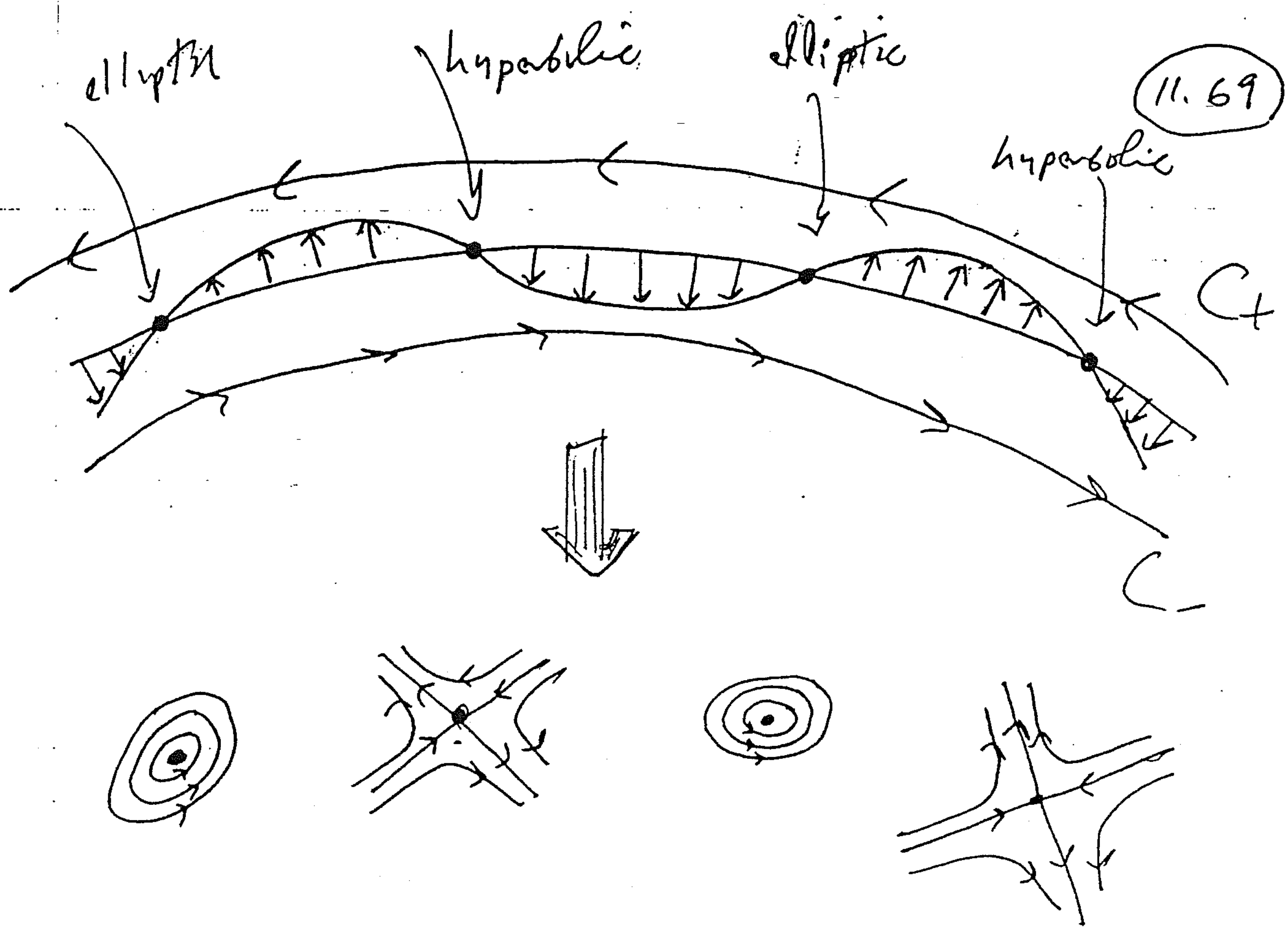
Hyperbolic fixed point at
$p = 0$
$\theta = \pi$

For the free particle:



Line of parabolic fixed points at
$p = 0$

Now we return to our discussion of the Poincare-Birkhoff theorem. We saw that T_ϵ^s has fixed points. Are these fixed points elliptic or hyperbolic?



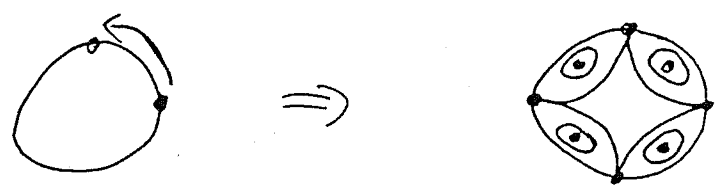
By considering the geometry of the orbits near the fixed points, we can see that the type of fixed point alternates as we travel along the radially mapped curve C_ϵ:

elliptic-hyperbolic-elliptic-hyperbolic- - - -

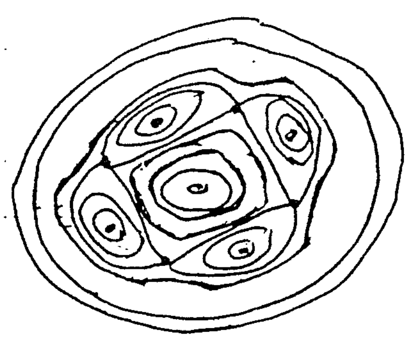
Thus, of the fixed points of T_ϵ^s, generically half are elliptic and half are hyperbolic.

If x_0 is an elliptic (or hyperbolic) fixed point of T_ϵ^s, then so is $T_\epsilon^m x_0$. This is why the number of fixed points is $2ks$ (k=integer), because both the number of elliptic fixed points and the number of hyperbolic fixed points is a multiple of s.

So if T rotates a circle C by the angle $2\pi \frac{r}{s}$, the orbits of the perturbed map near C break up into an "island chain" with ks islands (where typically $k=1$).



There are s elliptic fixed points, and T_ϵ carries a point near one of the fixed points to a point near a fixed point r steps along in the chain.
And there is a hyperbolic fixed point between each pair of elliptic fixed points.



Thus, the gap between the surviving invariant KAM Tori breaks up into smaller Tori. But we can now apply the KAM theorem and the Poincaré-Birkoff theorem to these smaller Tori. The irrational small tori are stable, but the small perturbations from the linear approximation to the map near the elliptic fixed point cause the rational small Tori to break up further into elliptic (and hyperbolic) fixed points surrounded by even smaller Tori. This process continues

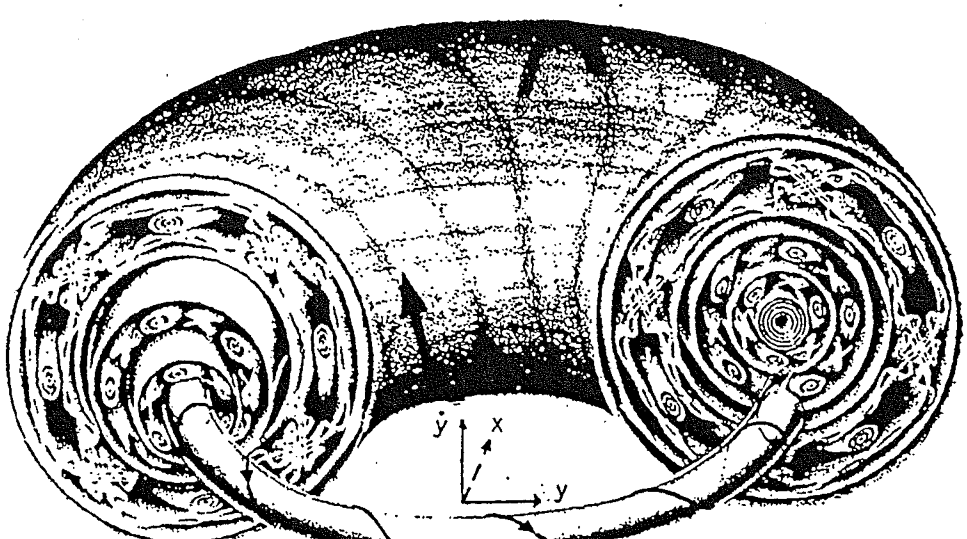


Figure 11
Nested K.A.M. tori in the 3-dimensional x,y,ẏ surface of constant energy of a Hamiltonian system like the Hénon-Heiles system (2.8-10); see figs. 10 and 4 (taken from [16]). Note the smooth regular K.A.M. tori about the center and about the elliptic orbits which, together with the chaotic regions, populate the gaps between the K.A.M. tori A magnification about such an elliptic orbit yields the same picture all over, *etc. ad infinitum*.

indefinitely, down to arbitrarily small scales. The resulting structure of the orbits in phase space -- with invariant tori of all sizes filling the gaps between larger invariant tori, is sketched on p. 11.71.

The "Homoclinic Tangle"

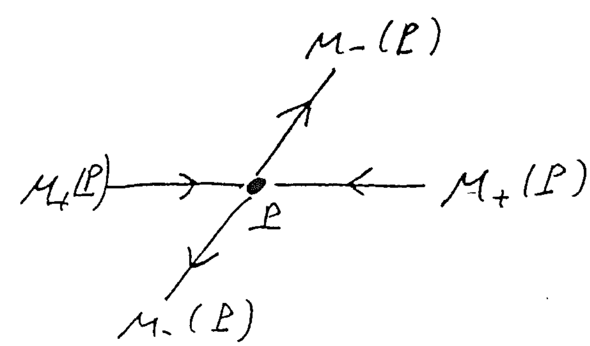
To complete the picture, we must understand the behavior of the orbits in the vicinity of the hyperbolic fixed points.

Associated with a hyperbolic fixed point $\underline{P}$ of a map T are the <u>stable</u> <u>manifold</u> $M_+(\underline{P})$ and the <u>unstable</u> manifold $M_-(\underline{P})$ of the fixed point. These are defined by

$$M_+(\underline{P}) = \{ X \mid \lim_{n\to\infty} T^n(X) = \underline{P} \},$$

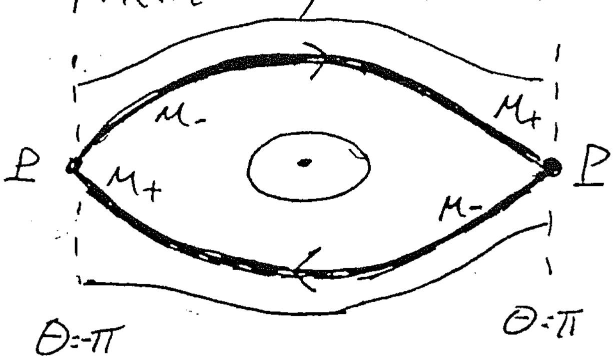
$$M_-(\underline{P}) = \{ X \mid \lim_{n\to -\infty} T^{-n}(X) = \underline{P} \}.$$

Points on the stable manifold approach $\underline{P}$ as the map is iterated forward in "time"; points on the unstable manifold approach $\underline{P}$ as the map is iterated backward in "time".



In the linearized approximation near the fixed point, for an area-preserving 2-dimensional map, the stable manifold is the line determined by the eigenvector with eigenvalue $|\lambda| < 1$, and the unstable manifold is the line determined by the eigenvector with eigenvalue $|\lambda| > 1$.

Now let's try to follow $M_+(P)$ and $M_-(P)$ beyond the region where the linearized approximation applies. For an integrable system --- the unstable manifold of one fixed point typically matches up smoothly with the stable manifold of another fixed point (or perhaps the same fixed point).

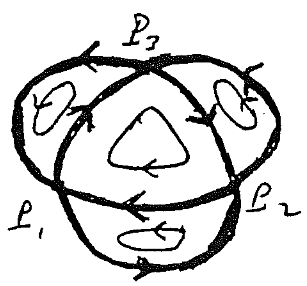


Recall, again, the pendulum.

Here $M_+(P) = M_-(P)$

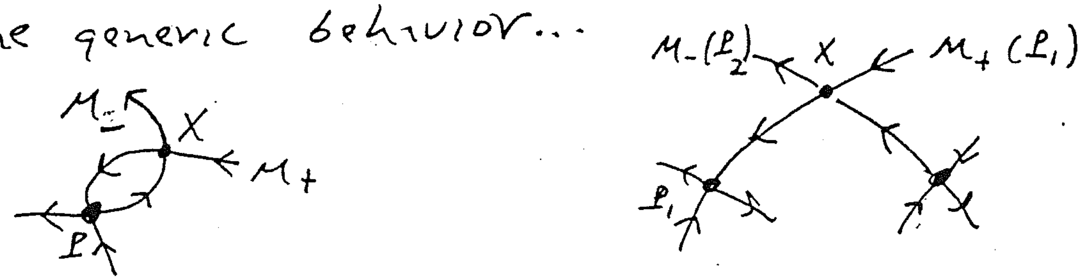
the orbit that leaves $P = (\theta = \pi, p = 0)$ in the unstable direction returns to P along stable direction.

More generally:



orbit that leaves one fixed point arrives at another fixed point.

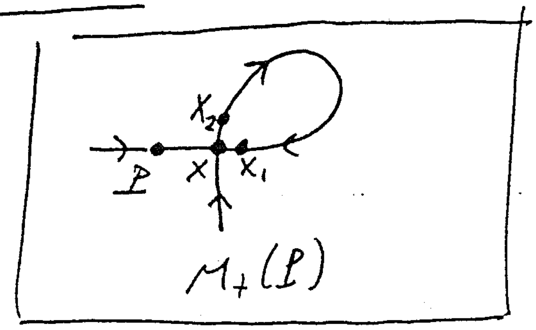
This smooth joining of stable and unstable manifolds is nongeneric. It does not occur in nonintegrable systems. The generic behavior...



is that the stable manifold of P and its unstable manifold cross one another at a point of intersection, called a "homoclinic point," or $M_+(P)$ intersects the unstable manifold of another fixed point, in which case the point of intersection is called a "heteroclinic point."

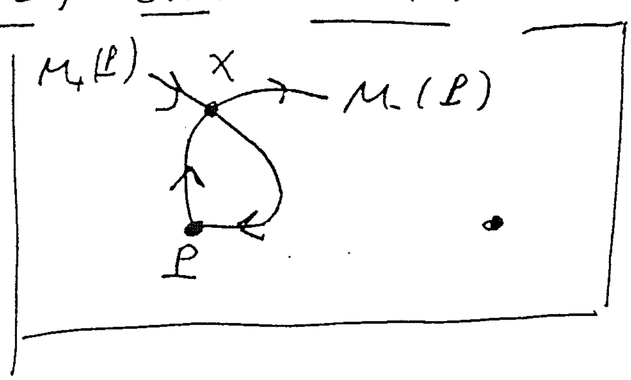
This simple observation leads to a remarkably complex picture of the structure of the orbits. We can show that $M_+(P)$ cannot intersect itself, nor can $M_-(P)$ intersect itself. But if $M_+(P)$ intersects $M_-(P)$ in one point (as it typically does) then there must be an infinite number of points of intersection! (The existence of one homoclinic point implies an infinity of others.) This means that there must be an exceedingly complex stretching, bending and folding of the phase space as the map is iterated -- in other words: the motion in the gaps between tori is chaotic.

$M_+(P)$ cannot intersect itself:



Suppose it does, at the point X. Then there are two points $X_{1,2}$ on $M_+(P)$ that are very close to X in phase space, but "far apart" on $M_+(P)$, as shown. After we iterate the map a few times, the images of X_1 and X_2 are close to P, but the image of the finite arc between X_1 and X_2 lies between the images of X_1 and X_2. This means that we can let X_1 and X_2 get arbitrarily close to one another, while their images stay a finite distance apart. This contradicts the continuity of the map, and so is not possible. Same goes for $M_-(P)$.

But $M_+(P)$ and $M_-(P)$ can intersect



This would not be possible for a continuous flow but it is allowed for a map.

If we interpret the diagram as a flow diagram, then the orbit beginning at X would not be unique. But for a map, the image of X does not have to be near X.

What we can say, though, is that if X is a homoclinic point, then so are TX and $T^{-1}X$.

Both of these points have the property that they approach P if T is iterated forward or backward, because X has that property.

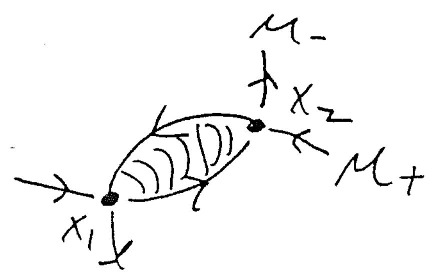
Furthermore, each point

$$T^n X, \quad -\infty < n < \infty$$

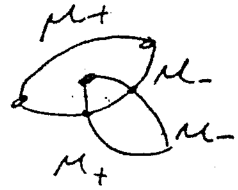
is a distinct point. (otherwise X would be a periodic point of T or T^{-1}, and then $T^n X$ could not approach P as $n \to \infty$ and $n \to -\infty$). So the $T^n X$ are an <u>infinite</u> number of homoclinic points that accumulate at P.

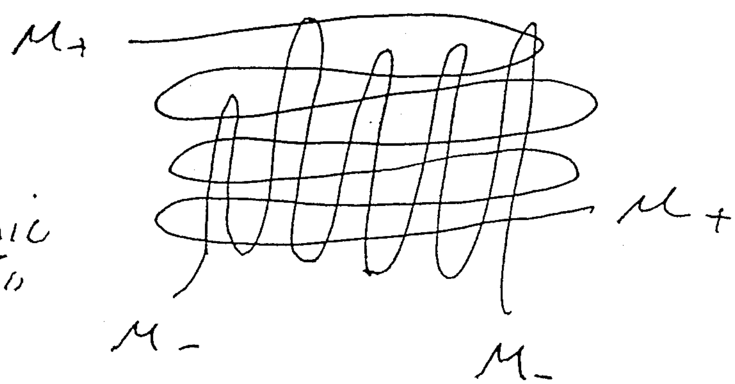
An even stronger statement: Between each pair of homoclinic points on $M_-(P)$ (or $M_+(P)$) is another homoclinic point. (so homoclinic points are actually <u>dense</u> in M_- and M_+).

To see this — suppose X_1 and X_2 are two successive homoclinic points on M_-. Consider the (shaded) area enclosed by the arcs of M_+ and M_-,



bounded by the points X_1 and X_2. Let T^n act on this region. Since T is area preserving, and only a finite area of phase space is available (intersection of $H=E$ manifold with Poincaré surface S is compact) image under T^n and T^m must overlap for some n and m. This means there is an intersection of M_+ and M_- in between X_1 and X_2, contrary to assumption.





= "The homoclinic tangle"

So M_+ and M_- must

stretch and fold, stretch and fold, ---

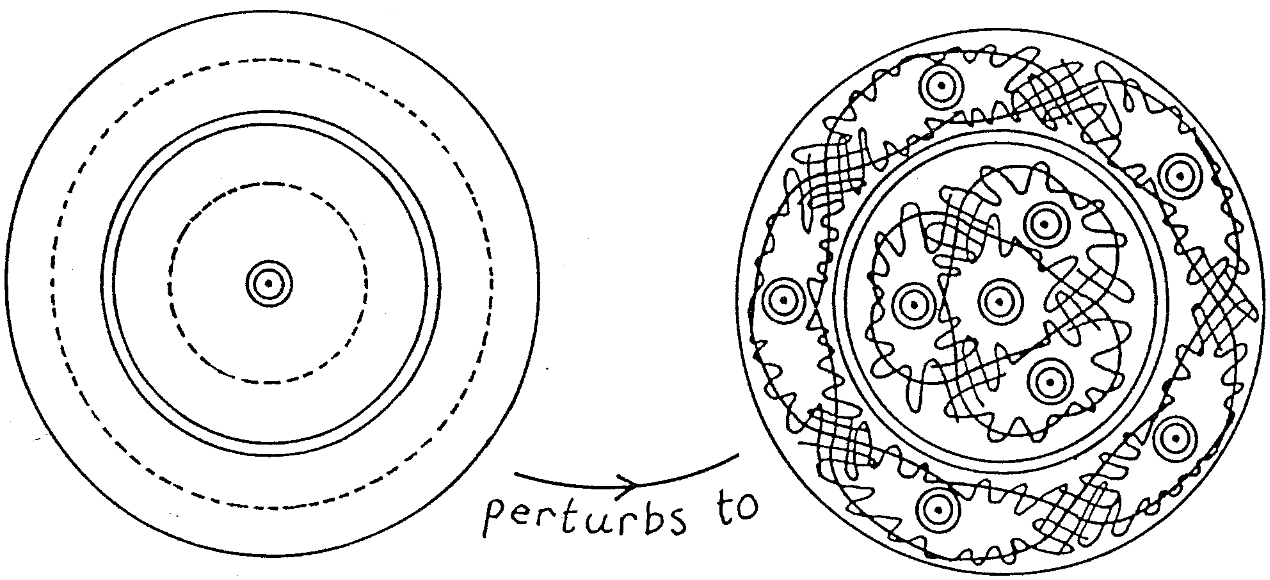
Orbits in the region of phase space containing the hyperbolic fixed points are area-filling, and do not lie on smooth invariant curves.

To summarize, for the generic perturbed integrable system, we have a wonderfully complex picture:

Each resonant torus breaks into elliptic and hyperbolic fixed points. The hyperbolic fixed points are surrounded by chaotic orbits. The elliptic fixed points are surrounded by invariant (irrational) closed curves, and also resonant (rational) curves that break up further, and repeat the whole structure ad infinitum.

— A truly astonishing compromise between "integrable" and "ergodic" behavior!

Note: for $N=2$, the surviving invariant tori are impenetrable strata that confine the chaotic orbits and prevent them from exploring the entire $H=E$ manifold. But for $N \geq 3$ this is not so. It is possible for a single chaotic orbit to densely fill all of the "gaps" between tori. This wandering of the orbits is called "Arnold Diffusion".



------ rational tori