Ph250a: Solutions to Homework 4

Problem 1.

The mode expansions of the fields are

$$j(z) = \sum_{n\in\mathbb{Z}} \frac{a_n}{z^{n+1}} \qquad \phi(z) = \sum_{n\in\mathbb{Z}} \frac{\phi_n}{z^{n+h}} \tag{1}$$

or reversing the Fourier expansion

$$a_n = \int_{|z|=\delta} \frac{dz}{2\pi i z} z^{n+1} j(z) \qquad \phi_n = \int_{|z|=\delta} \frac{dz}{2\pi i z} z^{n+h} \phi(z) \tag{2}$$

Using this expressions we have

$$[a_n, \phi_m] = \int_{|z|=\delta_1} \frac{dz}{2\pi i z} z^{n+1} j(z) \int_{|w|=\delta_2} \frac{dw}{2\pi i w} w^{m+h} \phi(w) - \int_{|w|=\delta_1} \frac{dw}{2\pi i w} w^{m+h} \phi(w) \int_{|z|=\delta_2} \frac{dz}{2\pi i z} z^{n+1} j(z) \tag{3}$$

where radial ordering (which is the time-ordering in radial quantisation) is implicit in these formulas ($\delta_1 > \delta_2$). Since the integrands are holomorphic outside of the line $z = w$ we can deform the first contour into the second one almost everywhere. The only difference is coming from the singularity of OPE near $z = w$

$$[a_n, \phi_m] = \int_{|z-w|=\delta_3} \int_{|w|=\delta_2} \frac{dz}{2\pi i z} \frac{dw}{2\pi i w} z^{n+1} w^{m+h} j(z)\phi(w) \tag{4}$$

$$\sim \int_{|z-w|=\delta_3} \int_{|w|=\delta_2} \frac{dz}{2\pi i z} \frac{dw}{2\pi i w} z^{n+1} w^{m+h} \left(\frac{q_\phi \phi(w)}{z-w} + \dots \right) \tag{5}$$

Only the simple pole of the OPE contributes to the integral over z

$$[a_n, \phi_m] = \int_{|w|=\delta_2} \frac{dw}{2\pi i w} w^{m+n+h} q_\phi \phi(w) = q_\phi \, \phi_{n+m} \tag{6}$$

Since $|\phi\rangle = \phi_{-h}|0\rangle$, we have

$$a_n|\phi\rangle = a_n \phi_{-h}|0\rangle = [a_n, \phi_{-h}]|0\rangle + \phi_{-h} a_n|0\rangle = q_\phi \phi_{n-h}|0\rangle \tag{7}$$

Therefore the condition of being primary is $a_n|\phi\rangle = 0$ if $n > 0$ and $a_0|\phi\rangle = q_\phi|0\rangle$.

Problem 2.

This problem can be either done in operator language or in state language. We will do it using both descriptions and we will ignore the antiholomorphic part

Let us start with the state language. By definition the Hilbert space of free-boson CFT is generated by the action of all operators a_n on the primary states $|P\rangle$ (which is defined as $a_n|P\rangle = 0$ for $n > 0$ and $a_0|P\rangle = |P\rangle$ where P is a real number and a_n is the mode expansion of $i\partial X$). Using the commutation relations $[a_n, a_m] = n\delta_{n+m,0}$ (which can be derived from the OPE of $i\partial X$ with itself) it is easy to convince yourself that the Hilbert space is spanned by

$$a_{-n_1}a_{-n_2}\dots a_{-n_k}|P\rangle \tag{8}$$

where n_i are positive integers. Therefore for all states with at least one action of a_{-n} there always exists a_n which acts non-zero on this state. Therefore primaries which must satisfy $a_n|\Phi\rangle = 0$ for $n > 0$ are given by $|P\rangle$. By state-operator correspondence they are related to e^{iPX}.

Now we will prove it in the operator language. By definition, the operators of the theory are generated by $\partial X(z)$, e^{iPX} their derivatives and results of the collision of this operators (i.e. taking the normal ordering of two operators). This can be motivated by path integral where all possible insertions are given by functions of $X(z)$ (which can be Fourier expanded) and derivatives of $X(z)$. Since the normal ordered pair of exponentials e^{ipX} and e^{iqX} at the same point is $e^{i(p+q)X}$, the operator which one can get are

$$: F(\partial X, \partial^2 X, \dots, \partial^k X)e^{iPX}(z) : \tag{9}$$

where F is a polynomial of the derivatives of $X(z)$. Since the OPE of $j(z) = i\partial X(z)$ with $\partial^k X(w)$ contains a non-vanishing pole of order $k+1$, the OPE of $j(z)$ with the operator (9) contains a non-vanishing pole of order at least 2 if the polynomial F is non-trivial. Therefore in order for the field to be primary with respect to $j(z)$ the polynomial must be constant. By direct computation we see that e^{iPX} is primary with charge P.