Week 5 (due Feb. 11)

1. Let E be a real vector bundle of rank r with a connection ∇. Let ω be the corresponding connection 1-form defined on some trivializing neighborhood U. As explained in class, ∇ enables us to define connections on vector bundles $\Lambda^k E$ for all k. In particular, we get a connection on a rank-one bundle $\Lambda^r E$. The corresponding connection 1-form is valued in ordinary real numbers (rather than matrices). Express this 1-form in terms of the matrix-valued form ω.

2. Let E be a real vector bundle equipped with a connection ∇ and a Euclidean metric h. Show that if ∇ is compatible with h, then horizontal transport preserves the norm of a section. Show that the converse is also true. That is, if parallel transport along any curve preserves the norm of any section, then ∇ is compatible with h.

3. Any coordinate system on $\mathbb{R}^2$ gives rise to a trivialization of $T\mathbb{R}^2$. By definition, the trivial connection on $T\mathbb{R}^2$ corresponds to zero connection 1-form ω in Cartesian coordinates (i.e. in the trivialization given by ∂_x and ∂_y).

(a) Find the connection 1-form on $T\mathbb{R}^2$ corresponding to the polar coordinates r, ϕ. Verify that the curvature tensor still vanishes.

(b) The vectors ∂_r and ∂_ϕ are orthogonal to each other, but only ∂_r has unit norm. So this is not an orthonormal trivialization. If we divide ∂_ϕ by its norm, we will get an orthonormal trivialization of $T\mathbb{R}^2$. Find the connection 1-form corresponding to this trivialization. Verify that the matrices ω^i_j are skew-symmetric and that the curvature tensor vanishes.