Week 5 (due May 6)

1. Consider a classical particle on a manifold X with an action

$$S = \int \left(p_i dq^i - \alpha_i dq^i - H(p,q)dt\right).$$

Here $\alpha_i(q)$ are components of a 1-form on X., and $H(p,q)$ is some function on X. The phase space is T^*X, but the symplectic form is not the canonical one.

(a) Determine the symplectic form by varying the action and paying attention to boundary terms. Write down the corresponding Poisson bracket for the coordinate functions p_i, q^i.

(b) Derive the equations of motion following from this action. Show that they are depend only on the closed 2-form $\beta = d\alpha$, so it makes sense to consider the situation where α is only locally well-defined. Show that in such a situation the symplectic form is still closed, but not necessarily exact. Interpret this system in physical terms, and in particular give a physical interpretation of the cohomology class of the 2-form β.

2. Consider a classical spin. This is a system whose phase space is S^2, and the symplectic form is the standard volume form $\omega = \sin\theta d\theta d\phi$.

(a) The group $SO(3)$ acts on this phase space and preserves ω. Determine the corresponding moment map (which should be a function on S^2 with values in the dual of the Lie algebra of $SO(3)$).

(b) Suppose the Hamiltonian is given by $H = \mu \cos\theta$. This corresponds to a spin in a magnetic field. Solve the equations of motion for this system.

3. Consider the space $\mathbb{C}^n$ with a symplectic form

$$i \sum_{k=1}^{n} dz^k \wedge d\bar{z}_k.$$

The group $U(n)$ acts on this symplectic space in an obvious manner:

$$z^i \mapsto g^i_j z^j, \quad \bar{z}_k \mapsto \bar{z}_l \left(g^{-1}\right)^l_k, \quad g \in U(n).$$

Determine the moment map for this action.