Week 4 (due April 29)

1. The Todd genus is the Chern genus corresponding to the analytic function $f(z) = z/(e^z - 1)$. Express the Todd genus in terms of Chern classes up to and including terms of cohomological degree 6.

2. Let S^2 be the 2-sphere with the standard Riemannian metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$.

(a) It is well known that using the stereographic projection one can parameterize the sphere minus a point by a complex coordinate z. So one can use z and $\bar{z}$ as complex coordinates on S^2. Compute the metric, the corresponding connection 1-form, and the curvature tensor of the tangent bundle of S^2 in terms of these coordinates. Note that TS^2 can be regarded as a complex rank-one bundle whose local trivialization is given by $\frac{\partial}{\partial z}$.

(b) Compute the connection 1-form for the spinor bundle Δ on S^2 and verify that the rank-two complex vector bundle Δ with its connection decomposes into a sum $\Delta_+ \oplus \Delta_-$, where $\Delta_+ \otimes \Delta_+$ is isomorphic to TS^2 (as a complex vector bundle with a connection), and $\Delta_- \otimes \Delta_-$ is isomorphic to its dual.

(c) Show that there are no nonzero harmonic spinors on S^2, in agreement with the index theorem.