1. (a) Gauge transformations take $A_\mu \to A_\mu + \partial_\mu f$, where f is a scalar function of spacetime. This means

$$\begin{aligned} F_{\mu\nu} &\to \partial_\mu A_\nu + \partial_\mu \partial_\nu f - \partial_\nu A_\mu - \partial_\nu \partial_\mu f \\ &= F_{\mu\nu}. \end{aligned} \tag{1}$$

Thus, the field strength is gauge invariant by itself, and we only need to worry about the Chern-Simons term. Then

$$\begin{aligned} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho &\to \epsilon^{\mu\nu\rho}(A_\mu + \partial_\mu f)\partial_\nu(A_\rho + \partial_\rho f) \\ &= \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \epsilon^{\mu\nu\rho} A_\mu \partial_\nu \partial_\rho f + \epsilon^{\mu\nu\rho} \partial_\mu f \partial_\nu A_\rho + \epsilon^{\mu\nu\rho} \partial_\mu f \partial_\nu \partial_\rho f \\ &= \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \epsilon^{\mu\nu\rho} \partial_\mu f \partial_\nu A_\rho. \end{aligned} \tag{2}$$

Also,

$$\int d^3x \epsilon^{\mu\nu\rho} \partial_\mu f \partial_\nu A_\rho = -\int d^3x \epsilon^{\mu\nu\rho} \partial_\nu \partial_\mu f A_\rho = 0 \tag{3}$$

after integration by parts, so we have

$$\int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \to \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \tag{4}$$

and we see that the action is gauge invariant. Now let's find the EoM for A using the Euler-Lagrange equation. First, we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial A_\mu} &= \frac{1}{2} k \epsilon^{\mu\nu\rho} \partial_\nu A_\rho \\ &= \frac{1}{4} k \epsilon^{\mu\nu\rho}(\partial_\nu A_\rho - \partial_\rho A_\nu) \\ &= \frac{1}{4} k \epsilon^{\mu\nu\rho} F_{\nu\rho}. \end{aligned} \tag{5}$$

Next, using

$$\frac{\partial F_{\rho\sigma}}{\partial(\partial_\nu A_\mu)} = \delta^\nu_\rho \delta^\mu_\sigma - \delta^\nu_\sigma \delta^\mu_\rho, \tag{6}$$

we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} &= -\frac{1}{2} \frac{\partial F_{\rho\sigma}}{\partial(\partial_\nu A_\mu)} F^{\rho\sigma} + \frac{1}{2} k \epsilon^{\rho\nu\mu} A_\rho \\ &= -\frac{1}{2}(F^{\nu\mu} - F^{\mu\nu}) + \frac{1}{2} k \epsilon^{\rho\nu\mu} A_\rho \\ &= F^{\mu\nu} + \frac{1}{2} k \epsilon^{\rho\nu\mu} A_\rho, \end{aligned} \tag{7}$$

which implies

$$\begin{aligned} \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} &= \partial_\nu F^{\mu\nu} + \frac{1}{2} k \epsilon^{\rho\nu\mu} \partial_\nu A_\rho \\ &= \partial_\nu F^{\mu\nu} - \frac{1}{2} k \epsilon^{\mu\nu\rho} \partial_\nu A_\rho \\ &= \partial_\nu F^{\mu\nu} - \frac{1}{4} k \epsilon^{\mu\nu\rho} F_{\nu\rho}. \end{aligned} \tag{8}$$

Putting the pieces together, we have that the equations of motion for the gauge field are

$$\begin{aligned}\frac{1}{4}k\epsilon^{\mu\nu\rho}F_{\nu\rho} &= \partial_\nu F^{\mu\nu} - \frac{1}{4}k\epsilon^{\mu\nu\rho}F_{\nu\rho}\\ \frac{1}{2}k\epsilon^{\mu\nu\rho}F_{\nu\rho} &= \partial_\nu F^{\mu\nu}\\ \epsilon^{\alpha\beta}_{\mu}\epsilon^{\mu\nu\rho}F_{\nu\rho} &= \frac{2}{k}\epsilon^{\alpha\beta}_{\mu}\partial_\nu F^{\mu\nu}\\ (\delta^{\alpha\nu}\delta^{\beta\rho} - \delta^{\alpha\rho}\delta^{\beta\nu})F_{\nu\rho} &= \frac{2}{k}\epsilon^{\alpha\beta}_{\mu}\partial_\nu F^{\mu\nu}\\ F^{\alpha\beta} &= \frac{1}{k}\epsilon^{\alpha\beta}_{\mu}\partial_\nu F^{\mu\nu}.\end{aligned} \tag{9}$$

This implies

$$\begin{aligned}F^{\alpha\beta} &= \frac{1}{k}\epsilon^{\alpha\beta}_{\mu}\partial_\nu\left(\frac{1}{k}\epsilon^{\mu\nu}_{\rho}\partial_\sigma F^{\rho\sigma}\right)\\ &= \frac{1}{k^2}\partial_\nu\partial_\sigma(\delta^{\alpha\nu}\delta^{\beta}_{\rho} - \delta^{\beta\nu}\delta^{\alpha}_{\rho})F^{\rho\sigma}\\ &= \frac{1}{k^2}(\partial^\alpha\partial_\sigma F^{\beta\sigma} - \partial^\beta\partial_\sigma F^{\alpha\sigma})\\ &= \frac{1}{k^2}[\partial^\alpha\partial_\sigma(\partial^\beta A^\sigma - \partial^\sigma A^\beta) - \partial^\beta\partial_\sigma(\partial^\alpha A^\sigma - \partial^\sigma A^\alpha)F^{\alpha\sigma}]\\ &= \frac{1}{k^2}(-\partial^\alpha\partial_\sigma\partial^\sigma A^\beta + \partial^\beta\partial_\sigma\partial^\sigma A^\alpha)\end{aligned} \tag{10}$$

The final line of (10) is equivalent to

$$\boxed{(k^2 + \partial^2)F_{\mu\nu} = 0,} \tag{11}$$

so we see that $F_{\mu\nu}$ satisfies the Klein-Gordon equation with mass $\boxed{m^2 = k^2.}$ Since we are in $2+1$ dimensions, these massive particles have $\boxed{1}$ polarization, which comes from the fact that A_0 is nondynamical and we can eliminate one more component through gauge transformations.

(b) The Gauss law constraint comes from the Euler-Lagrange equation for A_0. We have

$$\frac{\partial\mathcal{L}}{\partial A_0} = \frac{1}{2}k\epsilon^{0\nu\rho}\partial_\nu A_\rho = \frac{1}{2}kB. \tag{12}$$

Furthermore,

$$\begin{aligned}\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_0)} &= \partial_\nu F^{0\nu} - \frac{1}{4}k\epsilon^{0\nu\rho}F_{\nu\rho}\\ &= \partial_i F^{0i} - \frac{1}{2}kB\\ &= \nabla\cdot\vec{E} - \frac{1}{2}kB\end{aligned} \tag{13}$$

Then the analog of the Gauss law constraint is

$$\boxed{\nabla \cdot \vec{E} = kB.} \tag{14}$$

Now let's find the momentum conjugate to A_i. We have from (7)

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} = F^{i0} + \frac{1}{2}k\epsilon^{\rho 0 i}A_\rho, \tag{15}$$

which implies

$$\boxed{p^i = -E^i + \frac{1}{2}k\epsilon^{ij}A_j.} \tag{16}$$

Before we find the Hamiltonian, let's rewrite the Lagrangian a bit. First, notice that

$$F_{\mu\nu}F^{\mu\nu} = F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}, \tag{17}$$

but

$$F_{i0}F^{i0} = F_{0i}F^{0i} = -(\partial_i A_0 - \partial_0 A_i)^2 = -E^2 \tag{18}$$

and

$$F_{ij}F^{ij} = F_{12}F^{12} + F_{21}F^{21} = 2(\partial_1 A_2 - \partial_2 A_1)^2 = 2B^2. \tag{19}$$

Then we have

$$\mathcal{L} = -\frac{1}{2}(-E^2 + B^2) + \frac{1}{2}k(\epsilon^{0ij}A_0\partial_i A_j + \epsilon^{j0i}A_j\partial_0 A_i + \epsilon^{ij0}A_i\partial_j A_0). \tag{20}$$

The equation for the Hamiltonian is

$$\mathcal{H} = p^i\partial_0 A_i - \mathcal{L}. \tag{21}$$

We have

$$p^i\partial_0 A_i = -E^i\partial_0 A_i + \frac{1}{2}k\epsilon^{ij}A_j\partial_0 A_i, \tag{22}$$

so (21) implies

$$\mathcal{H} = -E^i\partial_0 A_i + \frac{1}{2}(-E^2 + B^2) - \frac{1}{2}k(\epsilon^{0ij}A_0\partial_i A_j + \epsilon^{ij0}A_i\partial_j A_0). \tag{23}$$

Integration by parts gives us

$$\epsilon^{ij0}A_i\partial_j A_0 \to -A_0\epsilon^{ij}\partial_j A_i = A_0 B. \tag{24}$$

Thus, we have

$$\mathcal{H} = -E^i\partial_0 A_i + \frac{1}{2}(-E^2 + B^2) - A_0 kB. \tag{25}$$

We can then write

$$
\begin{aligned}
\mathcal{H} &= -E^i(\partial_i A_0 - E_i) + \frac{1}{2}(-E^2 + B^2) - A_0 kB \\
&= E^2 + \frac{1}{2}(-E^2 + B^2) - E^i \partial_i A_0 - A_0 kB \\
&\to \frac{1}{2}(E^2 + B^2) + A_0(\partial_i E^i - kB)
\end{aligned}
\tag{26}
$$

We integrated by parts in the last step. Thus, we see that A_0 acts as a Lagrange multiplier which enforces (14), and assuming the Gauss's law constraint (i.e. plugging back in the equation of motion for A_0) gives us

$$
\boxed{\mathcal{H} = \frac{1}{2}(E^2 + B^2).}
\tag{27}
$$

I think this is actually what the problem wanted. Now, we can use (16) to write

$$
E^i = \frac{1}{2} k \epsilon^{ij} A_j - p^i,
\tag{28}
$$

which implies

$$
\begin{aligned}
E^2 &= \frac{1}{4} k^2 \epsilon^{ij} \epsilon_{ik} A_j A^k - k \epsilon^{ij} p_i A_j + p^2 \\
&= \frac{1}{4} k^2 A^2 - k \epsilon^{ij} p_i A_j + p^2,
\end{aligned}
\tag{29}
$$

so the action can be written

$$
\begin{aligned}
S &= \int d^3x \left[p^i \partial_0 A_i - \frac{1}{2}\left(p^2 - k\epsilon^{ij} p_i A_j + \frac{1}{4} k^2 A^2 + B^2 \right) - A_0(\partial_i E^i - kB) \right] \\
&= \int d^3x \left[p^i \partial_0 A_i - \frac{1}{2}\left(p^2 - k\epsilon^{ij} p_i A_j + \frac{1}{4} k^2 A^2 + B^2 \right) - A_0 \left(\frac{1}{2} kB - \partial_i p^i - kB \right) \right] \\
&= \int d^3x \left[p^i \partial_0 A_i - \frac{1}{2}\left(p^2 - k\epsilon^{ij} p_i A_j + \frac{1}{4} k^2 A^2 + B^2 \right) + A_0 \left(\partial_i p^i + \frac{1}{2} kB \right) \right].
\end{aligned}
\tag{30}
$$

(d) The Poisson brackets are given by

$$
\begin{aligned}
\{E_i(x), B(y)\} &= \{-p_i + \frac{1}{2}\epsilon_{ij} A_j, \epsilon_{kl} \partial_k A_l\} \\
&= -\epsilon_{kl} \partial_k \{p_i, A_l\} + \frac{1}{2} \epsilon_{ij} \epsilon_{kl} \partial_k \{A_j, A_l\} \\
&= -\epsilon_{kl} \partial_k \delta_{il} \delta^2(x-y) + \frac{1}{2} \epsilon_{ij} \epsilon_{kl} \partial_k (0) \\
&= \boxed{\epsilon_{ik} \partial_k \delta^2(x-y).}
\end{aligned}
\tag{31}
$$

(e) We can compute the propagator for the photon by writing down the Lagrangian in momentum space and inverting the coefficient of A^2. Because this theory is gauge invariant (see part (a)), we will need to gauge fix. We can do this by adding a gauge fixing term to the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - \frac{1}{2\xi}(\partial_\mu A^\nu)^2. \tag{32}$$

Then the Lagrangian in momentum space looks like

$$\mathcal{L} = -\frac{1}{2}\left[A_\mu\left(p^2g^{\mu\nu} - p^\mu p^\nu - \frac{1}{\xi}p^\mu p^\nu + ik\epsilon^{\mu\nu\rho}p_\rho\right)A_\nu\right]. \tag{33}$$

Defining

$$[\Delta^{-1}]^{\mu\nu}(p^2) = i\left[p^2g^{\mu\nu} - \left(1+\frac{1}{\xi}\right)p^\mu p^\nu + ik\epsilon^{\mu\nu\rho}p_\rho\right], \tag{34}$$

the most general form for $\Delta_{\mu\nu}(p^2)$ allowed by Lorentz invariance is

$$\Delta_{\mu\nu}(p^2) = -i(ag_{\mu\nu} + bp_\mu p_\nu + c\epsilon_{\mu\nu\rho}p^\rho), \tag{35}$$

Where a, b, and c are functions of p^2 and k. Note that this is possible because k has mass dimension one. Then we have

$$\begin{aligned}\Delta_{\mu\nu}[\Delta^{-1}]^{\nu\lambda} &= (ag_{\mu\nu} + bp_\mu p_\nu + c\epsilon_{\mu\nu\rho}p^\rho)\\ &\quad\times\left[p^2g^{\nu\lambda} - \left(1+\frac{1}{\xi}\right)p^\nu p^\lambda + ik\epsilon^{\nu\lambda\rho}p_\rho\right]\\ &= ap^2\delta^\lambda_\mu + \left[b - a - \frac{a}{\xi} - bp^2\left(1+\frac{1}{\xi}\right)\right]p_\mu p^\lambda\\ &\quad + (cp^2 + iak)\epsilon_\mu^{\lambda\rho}p_\rho + ick\epsilon_{\mu\nu\rho}\epsilon^{\nu\lambda\tau}p^\rho p_\tau\end{aligned} \tag{36}$$

But

$$\epsilon_{\mu\nu\rho}\epsilon^{\nu\lambda\tau}p^\rho p_\tau = p_\mu p^\lambda - p^2\delta^\lambda_\mu, \tag{37}$$

so we have

$$\begin{aligned}\Delta_{\mu\nu}[\Delta^{-1}]^{\nu\lambda} &= (a - ick)p^2\delta^\lambda_\mu + \left[b - a - \frac{a}{\xi} - bp^2\left(1+\frac{1}{\xi}\right) + ick\right]p_\mu p^\lambda\\ &\quad + (cp^2 + iak)\epsilon_\mu^{\lambda\rho}p_\rho.\end{aligned} \tag{38}$$

To start, this implies

$$\begin{aligned}cp^2 + iak &= 0\\ c &= -i\frac{ak}{p^2}.\end{aligned} \tag{39}$$

Furthermore, we must have

$$a - ick = \frac{1}{p^2}, \tag{40}$$

so we get

$$\begin{aligned} a - a\frac{k^2}{p^2} &= \frac{1}{p^2} \\ a &= \frac{1}{p^2 - k^2}. \end{aligned} \tag{41}$$

This also implies from (39) that

$$c = \frac{ik}{p^2(k^2 - p^2)}. \tag{42}$$

In the Lorenz gauge, $\xi \to 0$, so we should only worry about the terms with $\frac{1}{\xi}$. Then we have

$$\begin{aligned} \frac{a}{\xi} + bp^2\frac{1}{\xi} &= 0 \\ b &= -\frac{a}{p^2} \\ &= \frac{1}{p^2(k^2 - p^2)}. \end{aligned} \tag{43}$$

Altogether, the expression for the propagator is

$$\boxed{\Delta_{\mu\nu}(p^2) = \frac{-i}{p^2 + k^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - \frac{ik\epsilon_{\mu\nu\rho}p^\rho}{p^2}\right).} \tag{44}$$