Week 4 (due April 30)

1. Consider free scalar theory in four dimensions with zero mass. This theory is scale-invariant. Assume the space-time has Euclidean metric (i.e. we already performed Wick rotation).

(a) Compute the 2-point Green's function of the scalar field ϕ in coordinate space.

(b) Consider the operator product $\phi(x)\phi(0)$. Write down all operators of dimension up to 3 which can appear in the Operator Product Expansion (OPE). (Hint: first show that they must all be bilinear in ϕ by examining matrix elements of both sides of the OPE between suitable states).

(c) Compute the OPE coefficients for operators up to dimension 3. Hint: either compare the matrix elements of both sides of the OPE between suitable states in Fock space, or multiply both sides by a product $\phi(y)\phi(z)$ and compare the vacuum expectation values.

(d) Now consider the composite operator $O(x) = \phi^2(x)$. It is convenient to normal-order it, so that its vacuum expectation value is zero. This operator has dimension 2. Classify the operators which can appear in the OPE of $O(x)O(0)$ up to dimension 3. Compute the OPE coefficients of all these operators. Hint: it is convenient to think about insertions of O into vacuum expectation values as arising from adding to the action a new interaction $\int d^4x\psi(x)O(x)$ with some new classical field ψ and then expanding the Green's functions to the desired order in $\psi(x)$. For example, Green's functions with two insertions of $O(x)$ can be though of as arising from expanding Green's function in the modified theory to quadratic order in ψ. Then one can draw the usual Feynman diagrams in the modified theory to evaluate vacuum expectation values.

(e-f). Now add the interaction $L_{int} = \lambda\phi^4/4!$. Although the coupling λ is dimensionless, the theory is no longer scale-invariant, because of renormalization. If one uses dimensional regularization, the OPE coefficients now depend also on the $\bar{M}S$ scale μ. Consider the operator product $O(x)O(0)$ in this theory and compute the OPE coefficients of the first two operators which appear on the right-hand-side (i.e. 1 and $O(0)$) to order λ. (Hint: again, it is convenient to regard insertions of O as arising from adding a new interaction $\int d^4x\psi(x)O(x)$. Then to evaluate Green's functions with insertions of O we can use the Feynman diagrams of a modified theory which has two interaction: $\int d^4xL_{int}$ as well as $\int d^4x\psi(x)O(x)$, and therefore two kinds of vertices: one proportional to λ and one proportional to ψ. The main difference between the two interactions is that the number of λ vertices can be arbitrary, while the number of ψ vertices is fixed and determined by the number of O insertions.)