Week 1 (due Oct. 9)

Reading: Srednicki 1.1 (i.e. section 1 of Part 1) and Notes 1.

1. Consider the bosonic Fock space corresponding to the 1-particle Hilbert space $\mathcal{H}_1 = L^2(\mathbb{R}^3)$. Let $\Psi(x)$ and $\Psi^\dagger(x)$ be the standard annihilation and creation operator for bosonic particles. The particle number operator and the kinetic energy operator are defined by

$$\hat{N} = \int d^3x\, \Psi^\dagger(x)\Psi(x), \quad \hat{H}_0 = -\frac{1}{2m}\int d^3x\, \Psi^\dagger(x)\nabla^2\Psi(x).$$

(a) Show that $[\hat{H}_0, \hat{N}] = 0$. Compute the commutator $[\hat{N}, \Psi^\dagger(x)]$. What is the physical meaning of the result?

(b) Expand $\Psi(x)$ into Fourier series

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3} b(p)e^{ipx},$$

and express $\hat{N}$ and $\hat{H}_0$ in terms of $b(p)$ and $b^\dagger(p)$.

(c) Compute $[\hat{N}, b^\dagger(p)]$ and $[\hat{H}_0, b^\dagger(p)]$. What is the physical meaning of these results?

2. (a) Let H_1 be an infinite-dimensional one-particle Hilbert space. Consider a state in the bosonic Fock space $\mathcal{F}(H_1)$ which has the form

$$\exp\left(\sum_i a_i^\dagger \lambda_i\right)|0\rangle,$$

where λ_i are complex numbers, and $a_i^\dagger$ are creation operators with respect to some choice of basis in H_1. What conditions should λ_i satisfy in order for this state to have a finite norm? Compute the average (i.e. the expectation value of the) particle number operator in this state and the standard deviation of the particle number from the average.

(b) Consider modified bosonic creation and annihilation operators

$$b_i = a_i - \lambda_i, \quad b_i^\dagger = a_i^\dagger - \lambda_i^*.$$

Show that they satisfy the same commutation relations as $a_i, a_i^\dagger$ and that the state considered in part (a) is the vacuum state for $b_i, b_i^\dagger$.

(c) Consider a Hamiltonian in the bosonic Fock space $\mathcal{F}(H_1)$:

$$H = \sum_i \left(\omega_i a_i^\dagger a_i + \beta_i a_i + \beta_i^* a_i^\dagger \right),$$

where ω_i are positive numbers and β_i are complex numbers. This Hamiltonian does not commute with the particle number operator. Define new creation and annihilation operators b_i and $b_i^\dagger$ as in part (b) and choose λ_i so that H takes the form

$$H = E_0 + \sum_i \omega_i b_i^\dagger b_i,$$

where E_0 is a c-number. Since the particle number operators

$$N_i = b_i^\dagger b_i$$

have positive spectrum, this means that the ground state energy is E_0. Compute E_0 in terms of ω_i and β_i. It also follows from this that the ground state of H in terms of the original Fock space operators $a_i, a_i^\dagger$ and their vacuum has the form considered in part (a). Finally, it follows that the Hamiltonian H describes a system of free bosonic particles with energies ω_i, but their creation operators are $b_i^\dagger$ rather than $a_i^\dagger$.

(d) Let $a, a^\dagger$ be bosonic annihilation and creation operators, and t be a real number. Consider linear combinations

$$b = a \cosh t + a^\dagger \sinh t, \quad b^\dagger = a^\dagger \cosh t + a \sinh t.$$

Such a transformation is called a Bogolyubov transformation. Show that b and $b^\dagger$ satisfy the same commutation relations as a and $a^\dagger$, for any t. Find the vacuum state for the operators $b, b^\dagger$ (i.e. the state annihilated by b) as an element in the Fock space of a and $a^\dagger$. (N.B. Such a state is called a squeezed state).

(e) Consider the following Hamiltonian in the bosonic Fock space $\mathcal{F}(H_1)$:

$$H = \sum_i \left(\omega_i a_i^\dagger a_i + \frac{1}{2}\lambda_i a_i a_i + \frac{1}{2}\lambda_i a_i^\dagger a_i^\dagger \right),$$

where ω_i and λ_i are real numbers and $\omega_i > 0$ for all i. Find a Bogolyubov transformation such that in terms of $b_i, b_i^\dagger$ the Hamiltonian takes the form

$$E_0 + \sum_i \omega_i' b_i^\dagger b_i,$$

where E_0 and ω'_i are real numbers. This means that the energy of the ground state for H is E_0, and that H describes a system on noninteracting bosonic excitations with energies ω'_i. Express E_0 and ω'_i in terms of ω_i and λ_i.